**November 05** Lecture Schedule

Differential of composite function ([Takahashi, Kato] p.128, 4.4, [Kaneko] II p.19, 6.4, [Kodaira] II p.278, 6.2 e))

$f(x, y)$ : function of two variables $x$ and $y$.

$g(t), h(t)$: function of $t$.

Composite function is the function $f(g(t), h(t))$ we can obtain by the substitution.

The composite function $f(g(t), h(t))$ must be continuous if $f(x, y)$, $g(t)$, and $h(t)$ are all continuous.

Meaning.

Let differential $(a, b) = (g(c), h(c))$. We assume $g(t)$ ,and $h(t)$ are differential at $t = c$ , and $f(x, y)$ is (total) differntial at $(x, y) = (a, b)$. In such a case, the composite function $f(g(t), h(t))$ is differential at $t = c$ in which we can write as

$$\frac{d}{dt} f(g(t), h(t))|_{t=c} = \frac{\partial}{\partial x} f(a, b) g'(c) + \frac{\partial}{\partial y} f(a, b) h'(c).$$

Graphical implication: a tangent line is included in the tangent plane.

[Proof] Given our hypothesis, if $\sqrt{(x-a)^2 + (y-b)^2} \leq r$, function $k(r) \to 0$ that satisfies below must exist:

$$\begin{aligned} & \left| f(x, y) - f(a, b) - \frac{\partial}{\partial x} f(a, b)(x - a) - \frac{\partial}{\partial y} f(a, b)(y - b) \right| \\ \leq \ & k(r) \sqrt{(x-a)^2 + (y-b)^2} \end{aligned}$$

We let $F(t) = f(g(t), h(t))$, and which gives

$$\begin{aligned} & \left| F(t) - F(c) - \frac{\partial}{\partial x} f(a, b)(g(t) - g(c)) - \frac{\partial}{\partial y} f(a, b)(h(t) - h(c)) \right| \\ \leq \ & k(r) \sqrt{(g(t) - g(c))^2 + (h(t) - h(c))^2} \end{aligned}$$

Divide both sides of the equation by $t - c$, and further let $t \to c$then we can write as

$$\begin{aligned} & \left| \lim_{t \to c} \frac{F(t) - F(c)}{t - c} - \frac{\partial}{\partial x} f(a, b) g'(c) - \frac{\partial}{\partial y} f(a, b) h'(c) \right| \\ \leq \ & k(r) \sqrt{(g'(c)^2 + h'(c)^2} \to 0. \end{aligned}$$

The mean value theorem:

We apply the mean value theorem $F(1) - F(0) = F'(t)$ to the function $F(t) = f(a + t(x - a), b + t(y - b))$:

$$
\begin{aligned}
& f(x, y) - f(a, b) \\
= \; & \frac{\partial}{\partial x} f(a + t(x - a), b + t(y - b))(x - a) + \frac{\partial}{\partial y} f(a + t(x - a), b + t(y - b))(y - b).
\end{aligned}
$$

Higher-order derivatives:

The second-order partial derivatives $f_{xx}(x, y)$ and $f_{xy}(x, y)$ can be defined if and only the partial derivative $f_x(x, y)$ is partial differentiable. $f_{yx}(x, y)$ and $f_{yy}(x, y)$ can be treated in the same way. The higher-order partial derivatives are defined the same way.

Notation. $\frac{\partial^2}{\partial x^2} f(x, y)$, $\frac{\partial^2}{\partial x \partial y} f(x, y)$, and so on.