October 15

Definition. $\lim_{(x,y)\to(a,b)} f(x,y) = A$ implies the existence of function $g(r)$ for $r > 0$, which stands for the characteristics (1) and (2) as follows:

(1) $|f(x,y) - A| \le g(r)$ if $0 < \sqrt{(x-a)^2 + (y-b)^2} \le r$.

(2) $\lim_{r\to+0} g(r) = 0$.

(In ε-δ proof, r becomes δ, and $g(r)$ becomes ε.)

Explanation. $\lim_{(x,y)\to(a,b)} f(x,y) = A$ implies that we can write $|f(x,y) - A|$ when $(x,y) \to (a,b)$, and $(x,y) \to (a,b)$ represents $\sqrt{(x-a)^2 + (y-b)^2} \to 0$.

If we let the maximum value of $|f(x,y) - A|$ in the range of $\sqrt{(x-a)^2 + (y-b)^2} \le r$ be $M(r)$, which allows us to write $M(r) \to 0$ at $r \to 0$. There is no guarantee that the maximum value exists, and there is no need for finding the particular values of maximum. Characteristics of $M(r)$: $|f(x,y) - A| \le M(r)$ if $0 < \sqrt{(x-a)^2 + (y-b)^2} \le r$.

We say $f(x,y)$ is continuous on (a,b) when $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. If $f(x,y)$ is continuous on each point of U, we call $f(x,y)$ is continuous on U. If we can clarify U, then we can simply call that $f(x,y)$ is being continuous.

Differentiation of the two variables function: ([Nagase, Ashino] 7.3, [Kaneko] II 6.2, [Kodaira] II §6.2)

We assume the partial derivative y being held constant and differentiate with respect to x. Take $y = b$, we consider the function $f(x,b)$ of the single variable x. If we can define $\lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$, which differentiates $f(x,b)$ with respect to x, we call $f(x,y)$ partial differentiable at $(x,y) = (a,b)$ with respect to x. The value obtained by such differentiation is called a partial differential coefficient of

$f(x,y)$ at $(x,y) = (a,b)$ with respect to x,

and expressed as $\frac{\partial f}{\partial x}(a,b)$.

Likewise, we can define the partial differential coefficient

$\lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k} = \frac{\partial f}{\partial y}(a,b)$

of $f(x,y)$ at $(x,y) = (a,b)$ with respect to y.

Other notations $f_x(a,b), f_y(a,b)$ and etc.

Note that being partial differentiable does not necessary indicate the continuity characteristics as we explained in above.

Total differential: What is the correct definition of the differential calculus? Differential calculus can be understood as the first-order approximation. What $f(x, y)$ being (total) differentiable at $(x, y) = (a, b)$ means is that there is a first-order equation that approximates accurately.

In other words,

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - (\frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - c) + f(a, b))}{\sqrt{(x - a)^2 + (y - b)^2}} = 0$$

The plane which can be determined by
$z = \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - c) + f(a, b)$
is called the tangent plane.