July 9 Lecture Schedule

Improper integral ([1] p.90-94, [2] p.120-122, [3] p.175-189)

We define an improper integral $\int_a^b f(x)dx$ of function $f(x)$ as $\lim_{t\to a}\int_t^b f(x)dx$ which is continuous on a half-open interval $(a,b]$. In the same way, we define the improper integral for $[a,b)$. Now we assume $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for (a,b) and consider convergence of each integral in which the improper integral to be defined by sum of the two integrals. Example.

$$\int_0^1 x^a dx = \begin{cases} \frac{1}{a-1} & a > 1, \\ \infty & a \leq 1, \end{cases}$$

$\int_0^1 x^a dx = \lim_{t\to 0}[\frac{x^{a+1}}{a+1}]_t^1 = \frac{1}{a+1} - \lim_{t\to 0}\frac{t^{a+1}}{a+1}$ (at $a \neq -1$).

$\int_0^1 x^{-1}dx = \lim_{t\to 0}[\log x]_t^1 = -\lim_{t\to 0}\log t$.

Convergence test using majorant function: if $|f(x)| \leq g(x)$ and $\int_a^b g(x)dx$ are convergent, then $\int_a^b f(x)dx$ must be absolutely convergent.

Now, we let an infinite interval $f(x)$ be the continuous function defined on $[a,\infty)$.

We define an improper integral $\int_a^\infty f(x)dx$ as $\lim_{t\to\infty}\int_a^t f(x)dx$, and in the same way, we may define for (a,∞), $(-\infty,\infty)$.

Example.

$$\int_1^\infty x^{-a} dx = \begin{cases} \frac{1}{a-1} & a > 1, \\ \infty & a \leq 1, \end{cases}$$

$\int_1^\infty x^{-a}dx = \lim_{t\to\infty}[\frac{x^{-a+1}}{-a+1}]_1^t = \lim_{t\to\infty}\frac{t^{-a+1}}{-a+1} + \frac{1}{a-1}$ (where $a \neq 1$).

$\int_1^\infty x^{-1}dx = \lim_{t\to\infty}[\log x]_1^t = \lim_{t\to\infty}\log t$.

Comparison with series convergence: ([1] pp.182-183 Theorem 4, [2] p.126, [3] p.207 Theorem 5.2)

Let $f(x) \geq 0$ be the continuous monotone decreasing function defined on $x \geq 1$. A sequence $a_1, a_2, \ldots$ is defined by $a_n = f(n)$.

Here, in order for a series $\sum_{n=1}^\infty$ to converge, it is a necessary and sufficient condition that the improper integral $\int_1^\infty f(x)dx$ converges. We can write

$$\int_1^\infty f(x)dx \leq \sum_{n=1}^\infty a_n \leq \int_1^\infty f(x)dx + a_1$$

.

Thus, $\int_1^n f(x)dx \leq \sum_{k=1}^n a_k \leq \int_1^n f(x)dx + a_1$ is given, and we know from which that above follows.

Example: $\sum_{n=1}^{\infty} \frac{1}{n^k}$ diverges when $k = 1$ while which converges when $k \geq 2$. We can wirte $\frac{1}{k-1} \leq \sum_{n=1}^{\infty} \frac{1}{n^k} \leq \frac{k}{k-1}$ if $k \geq 2$.

$\int_1^{\infty} \frac{1}{x^k} dx$ diverges when $k = 1$ while which gives $\int_1^{\infty} \frac{1}{x^k} dx = \frac{1}{k-1}$ when $k \geq 2$.

Additional notes on approximation by polynomials.

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$$\frac{f(x) - p(x)}{x^n} \to 0, \ \frac{g(x) - q(x)}{x^n} \to 0 \ (x \to 0)$$

, we can write

$$\frac{f(x)g(x) - p(x)q(x)}{x^n} = \frac{f(x)(g(x) - q(x))}{x^n} + \frac{q(x)(f(x) - p(x))}{x^n} \to 0 \ (x \to 0)$$

. Likewise, if we let

$$\frac{f(x) - a(1 - xp(x))}{x^n} \to 0 \ (x \to 0)$$

at $a \neq 0$, we can express

$$\begin{aligned} & \frac{1}{f(x)} - \frac{1}{a} \frac{1 - x^{n+1}p(x)^{n+1}}{1 - xp(x)} = \frac{a(1 - xp(x)) - f(x)(1 - x^{n+1}p(x)^{n+1})}{af(x)(1 - xp(x))} \\ = & \frac{-(f(x) - a(1 - xp(x))) + x^{n+1}f(x)p(x)^{n+1}}{af(x)(1 - xp(x))} \end{aligned}$$

. Thus,

$$\frac{1}{x^n}\left(\frac{1}{f(x)} - \frac{1}{a}\left(1 + xp(x) + x^2p(x)^2 + \cdots + x^np(x)^n\right)\right) \to 0 \ (x \to 0)$$

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