June 25 Lecture Schedule

Definite integral ([1] p.82, [2] p.101-104, 127-133, [3] p.153-158)

We let $f(x)$ be a continuous function defined on a closed interval $[a, b]$. We also define the definite integral $\int_a^b f(x)dx$ as follows. Suppose $\Delta : a = a_0 \le a_1 \le \cdots \le a_{i-1} \le a_i \le \cdots \le a_{n-1} \le a_n = b$ is a partition of the closed interval $[a, b]$. Take x_i to satisfy $a_{i-1} \le x_i \le a_i$ for each $i = 1, \ldots, n$, and consider the sum $S = \sum_{i=1}^n f(x_i)(a_i - a_{i-1})$. The limit of S exists when the diameter of the partition $|\Delta| = \max_i(a_i - a_{i-1})$ is approached to 0, where we define the limit $\int_a^b f(x)dx$.

For each $i = 1, \ldots, n$, we let m_i be the minimum value and M_i be the maximum value of $f(x)$ at $a_{i-1} \le x \le a_i$. If we let $s_\Delta = \sum_{i=1}^n m_i(a_i - a_{i-1})$ and $S_\Delta = \sum_{i=1}^n M_i(a_i - a_{i-1})$, we can write $s_\Delta \le S \le S_\Delta$ so that giving $S_\Delta - s_\Delta = \sum_{i=1}^n (M_i - m_i)(a_i - a_{i-1}) \le \max_i(M_i - m_i)(b - a)$.

Now, we let Δ' be another partition of the interval. If Δ'' is a common refinement of Δ and Δ' then, we may express $s_\Delta \le s_{\Delta''} \le S_{\Delta''} \le S_\Delta$ and $s_{\Delta'} \le s_{\Delta''} \le S_{\Delta''} \le S_{\Delta'}$. To show the existence of a limit, we need to demonstrate $\max_i(M_i - m_i) \to 0$ at $|\Delta| \to 0$.

In this case, note that we assume $f(x)$ to be continuously differentiable for the simplicity of calculations.

Let M' be the maximum value of $|f'(x)|$ at $a \le x \le b$, in which provides $M_i - m_i \le M'(a_i - a_{i-1})$. Thus, $\max_i(M_i - m_i) \le M'|\Delta| \to 0$ (when $|\Delta| \to 0$.)
Example,

$$\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \log 2$$

We take $f(x) = \frac{1}{x}$ と and define the partition of the interval $[1, 2]$ as $1 = \frac{n}{n} \le \frac{n+1}{n} \le \cdots \frac{n+i-1}{n} \le \frac{n+i}{n} \le \cdots \le \frac{2n}{n} = 2$ so that we let $x_i = \frac{n+i}{n}$. Here, the sum $S = \sum_{i=1}^n f(x_i)(a_i - a_{i-1})$ can be written as $\sum_{i=1}^n (\frac{n+i}{n})^{-1}\frac{1}{n} = \sum_{i=1}^n \frac{1}{n+i}$. Thus, the limit is given by $\lim_{n\to\infty}\sum_{i=1}^n \frac{1}{n+i} = \int_1^2 \frac{1}{x}dx = [\log x]_1^2 = \log 2$..

Fundamental theorem of differential and integral calculus ([1] p.85 Theorem 9, [2] p.101, [3] p.165 Theorem 4.4)

$$\frac{d}{dx}\int_a^x f(x)dx = f(x)$$

The left side of the equation represents the common values of $\lim_{h\to+0}\frac{1}{h}\int_x^{x+h} f(x)dx$ and $\lim_{h\to+0}\frac{1}{h}\int_{x-h}^x f(x)dx$. If we let the maximum values and the minimum values of $f(y)$ be M_h and m_h at $x-h \leq y \leq x+h$, which allows us to write as

$$m_h h \leq \int_x^{x+h} f(x)dx, \int_{x-h}^{h} f(x)dx \leq M_h h.$$

Divide the both sides of the equation by h and let $h \to 0$ then we obtain $m_h \to f(x)$,and $M_h \to f(x)$.