June 18 Lecture Schedule

Alternating series convergence ([1] p.188 Theorem 8, [2] p.24 Theorem 1.4, [3] p.45 Theorem 1.23)

$\sum_{n=0}^{\infty}(-1)^n a_n$ converges if it is monotonically decreasing and $a_n > 0, \lim_{n\to 0} a_n = 0$.

$$|a - \sum_{k=0}^{n}(-1)^k a_k| \leq |a_{n+1}|.$$

Power series expansion for other functions. ([1] p.44 Example 20(5), p.194 Exercise 3, [2] p.73, [3] p.239 Example 5.11)

$$\begin{aligned}
& (1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n \\
= \; & 1 + ax + \frac{a(a-1)}{2}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \frac{a(a-1)(a-2)(a-3)}{4!}x^4 + \cdots . \\
& \frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2^n n!} x^{2n}, \\
= \; & 1 + \frac{1}{2}x^2 + \frac{3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots .
\end{aligned}$$

([1] p.195 Problem 11(3), [2] p.75, [3] p.239 Example 5.14)

$$\begin{aligned}
\text{Arcsin}x &= \int_0^x \frac{1}{\sqrt{1-x^2}}dx = \int_0^x \left(\sum_{n=0}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n} x^{2n} \right) dx \\
&= \sum_{n=0}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n\cdot(2n+1)} x^{2n+1} \\
&= x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \cdots .
\end{aligned}$$

Differential equation: ([1] p.197, [2] p.134.)

The equations which include $f(x)$ and $f'(x)$ (or higher order derivatives.)

Solving the differential equation: find the function $f(x)$ which satisfies the equation.

Example. $f'(x) = f(x)$. Solution is $f(x) = Ce^x$(C is held constant).

$$\left(\frac{f(x)}{e^x}\right)' = \frac{e^x f'(x) - e^x f(x)}{e^{2x}} = \frac{f'(x) - f(x)}{e^x} = 0$$

Thus, $\dfrac{f(x)}{e^x}$ is held constant. In general, $f(x)$ cannot be defined uniquely by the equation.

Initial condition: arrange some values such as $f(a)$,and $f'(a)$.

Example. Equation $f'(x) = f(x)$, initial condition $f(0) = 1$; solution is $f(x) = e^x$.

Another proof of $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ stands for $f'(x) = f(x)$,and $f(0) = 1$.

Defined on $-1 < x < 1$, $(1+x)^a$ satisfies the initial condition $f(0) = 1$, in which the only existing solution for the differential equation (1+x)f'(x)=af(x).

$$\left(\frac{f(x)}{(1+x)^a}\right)' = \frac{(1+x)^a f'(x) - a(1+x)^{a-1} f(x)}{(1+x)^{2a}} = \frac{(1+x)f'(x) - af(x)}{(1+x)^{a+1}} = 0$$

Thus, we know that $\dfrac{f(x)}{(1+x)^a}$ is held constant.

Given $|\binom{a}{n+1}/\binom{a}{n}| = |\frac{a-n}{n+1}| \to 1$, the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} \binom{a}{n} x^n$ is 1.

So, the function $f(x) = \displaystyle\sum_{n=0}^{\infty} \binom{a}{n} x^n$ is defined on $-1 < x < 1$.

This satisfies the initial condition $f(0) = 1$, thus we can write

$$f'(x) = \sum_{n=0}^{\infty} \binom{a}{n} n x^{n-1}.$$

$n\binom{a}{n} = a\binom{a-1}{n-1}$, $\binom{a-1}{n-1} + \binom{a-1}{n} = \binom{a}{n}$ provides

$$(1+x)f'(x) = \sum_{n=0}^{\infty} (a\binom{a-1}{n-1} + a\binom{a-1}{n})x^n = a\sum_{n=0}^{\infty} \binom{a}{n} x^n = af(x).$$

Thus, we can write

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$$