クレジット：
Mathematics and Informatics Center 文科系のための線形代数・解析Ⅰ
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# 第12講 行列の対角化

## 12-1 行列式とは (復習)

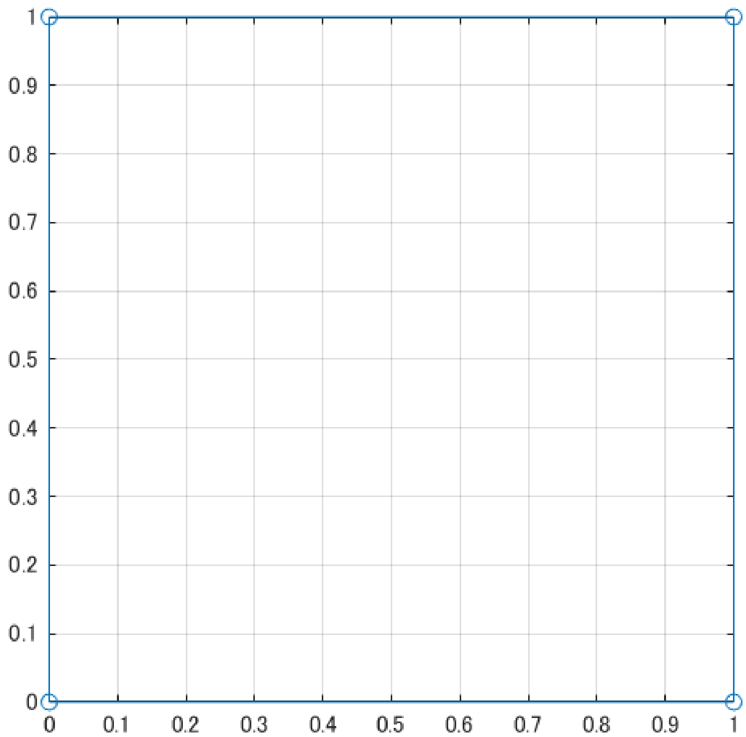
行列の線形写像としての大事な情報を含んでいる量

変換後の図形の面積・体積は「行列式の絶対値」倍になる

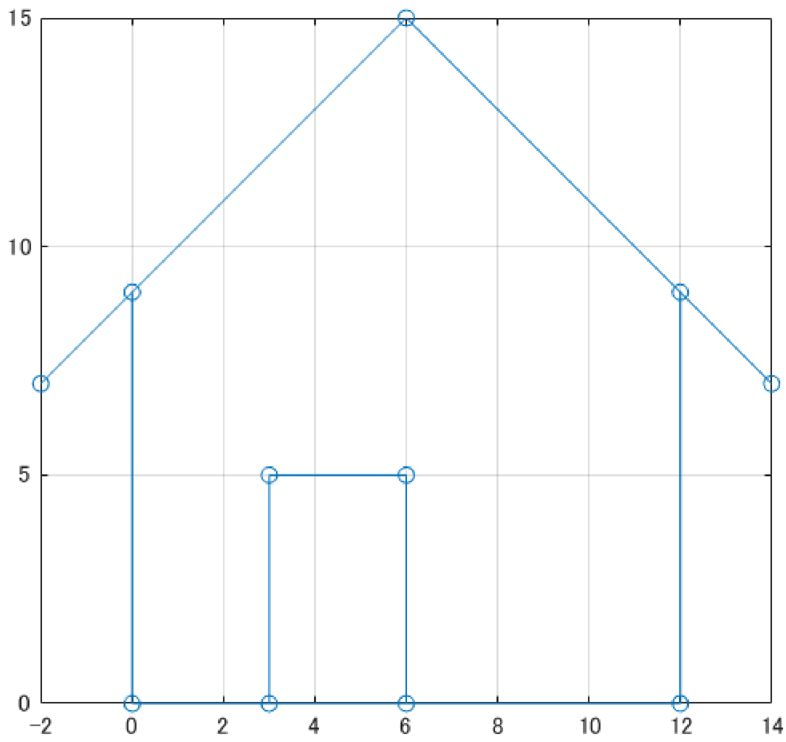
行列式が負の場合、図形は裏返る

以下の図形の変換を考える

```
S = [0 1 1 0 0; 0 0 1 1 0];
plot(S(1,:), S(2,:),"o-")
daspect([1 1 1])
grid on
```




```
H = [0 0 -2 6 14 12 12 3 3 6 6 0;
     0 9 7 15 7 9 0 0 5 5 0 0];
plot(H(1,:), H(2,:),"o-")
daspect([1 1 1])
grid on
```




行列 $A = \begin{pmatrix} 0.9 & 0.6 \\ 0.3 & 0.8 \end{pmatrix}$ による線形変換

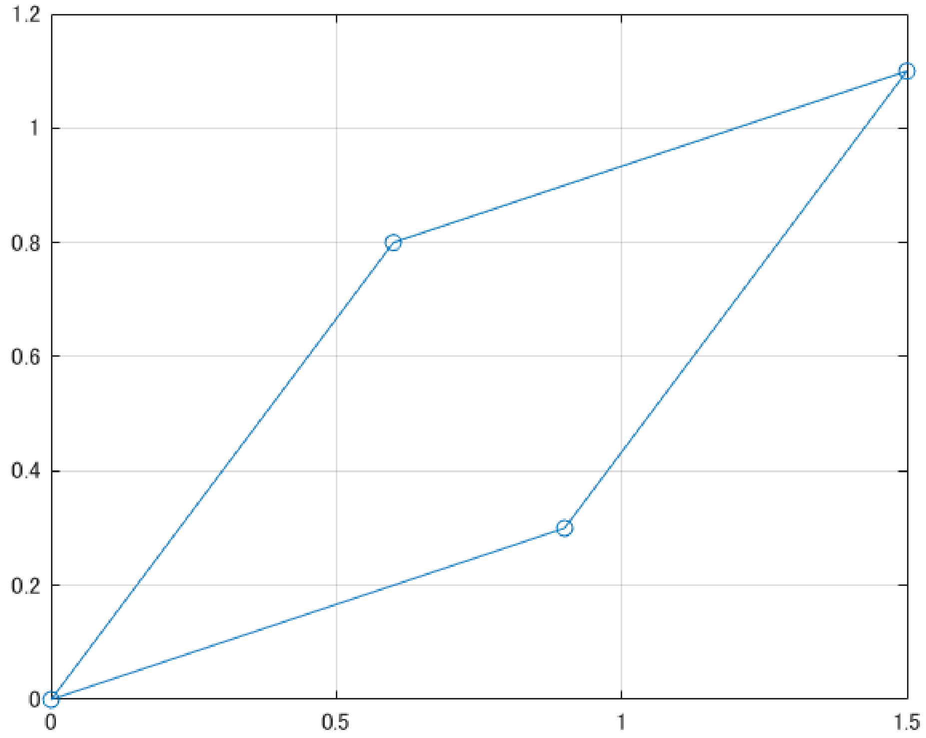
```
A = [ 0.9 0.6; 0.3 0.8]
```

```
A = 2×2
    0.9000    0.6000
    0.3000    0.8000
```

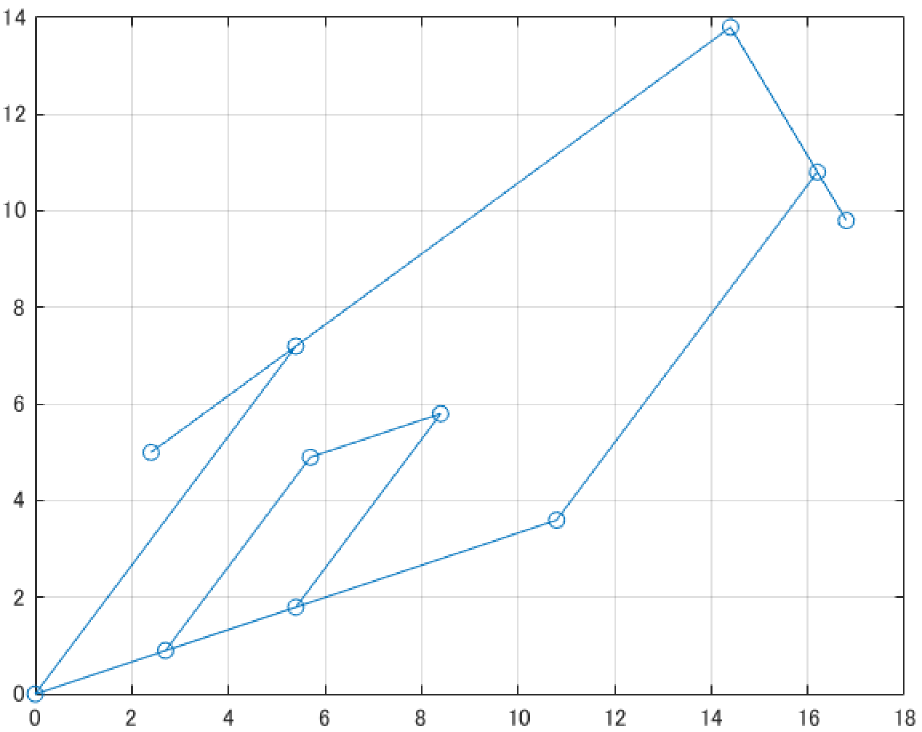
```
AS = A * S
```

```
AS = 2×5
         0    0.9000    1.5000    0.6000         0
         0    0.3000    1.1000    0.8000         0
```

```
plot(AS(1,:), AS(2,:),"o-")
daspect([1 1 1])
grid on
```




```
AH = A * H;
plot(AH(1,:), AH(2,:),"o-")
daspect([1 1 1])
grid on
```




```
det(A)
```

```
ans = 0.5400
```

変換により正方形は平行四辺形に, またその面積は1から

```
0.6*0.8/2 + 0.9*(0.8+1.1)/2 - 0.9*0.3/2 - 0.6*(0.3+1.1)/2
```

```
ans = 0.5400
```

に変化. この量は上の行列式の値に等しい.

一般の $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ の場合, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ は $\begin{pmatrix} a \\ c \end{pmatrix}$ に, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ は $\begin{pmatrix} b \\ d \end{pmatrix}$ に移る. その面積は

```
syms a b c d
expand(b*d/2 + a*(d+c+d)/2 - a*c/2 - b*(c+c+d)/2)
```

ans = $ad - bc$

## 12-2 行列式の性質

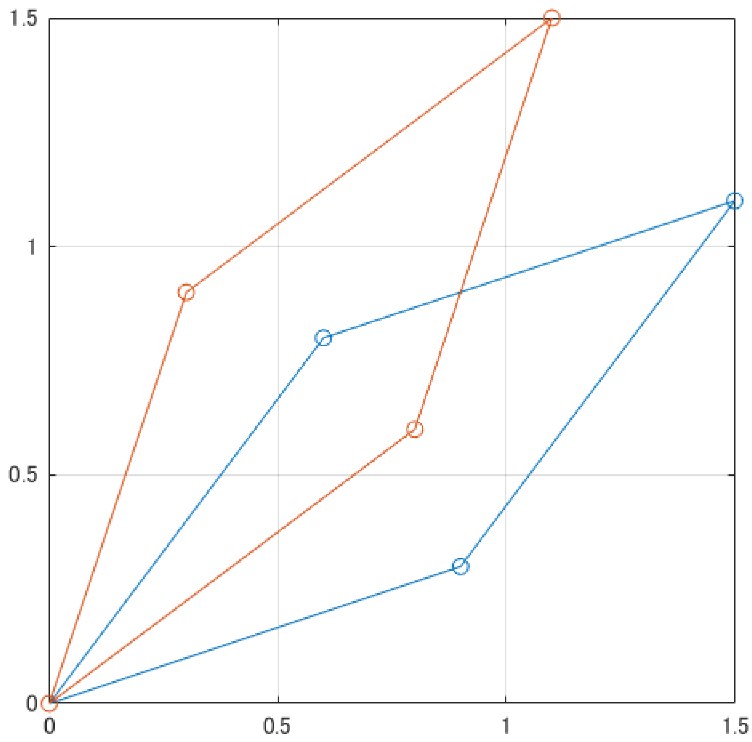
### 性質1: 行の交換

行を交換すると変換後の x 軸と y 軸が入れ替わる. 面積は不変だが, 図形は裏返る

```
B = [0.3 0.8; 0.9 0.6];

BS = B * S
```

```
BS = 2×5
         0    0.3000    1.1000    0.8000         0
         0    0.9000    1.5000    0.6000         0
```

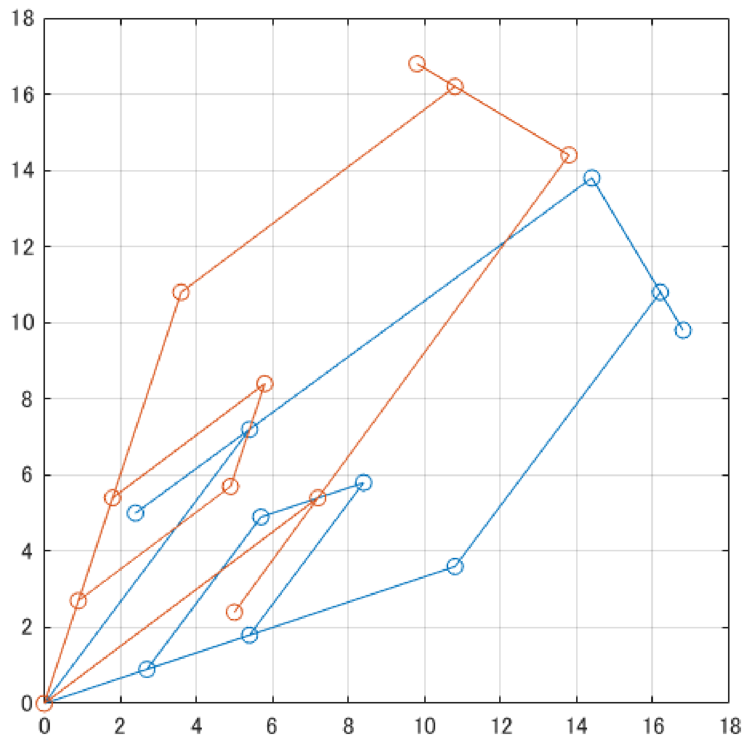
```
plot(AS(1,:), AS(2,:),"o-")
hold on
plot(BS(1,:), BS(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
BH = B * H
```

```
BH = 2×12
         0    7.2000    5.0000   13.8000    9.8000   10.8000    3.6000    0.9000    4.9000
         0    5.4000    2.4000   14.4000   16.8000   16.2000   10.8000    2.7000    5.7000
```

```
plot(AH(1,:), AH(2,:),"o-")
hold on
plot(BH(1,:), BH(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
det(B)
```

```
ans = -0.5400
```

## 性質1: 列の交換

二つのベクトルが入れ替わる. 平行四辺形は不変だが, 家は裏返る

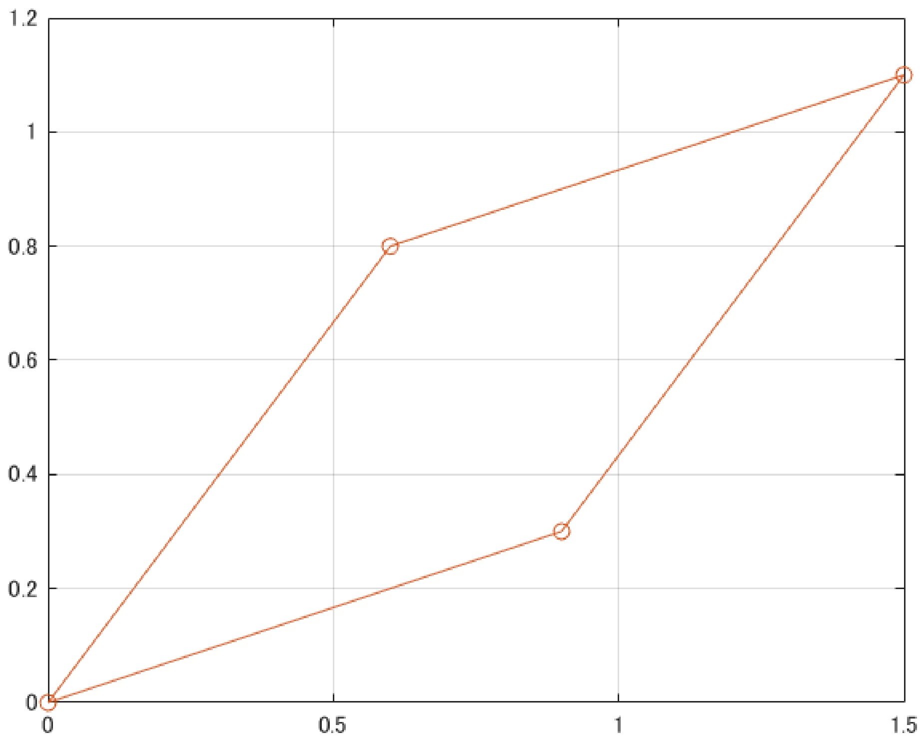
```
B = [0.6 0.9; 0.8 0.3]
```

```
B = 2×2
    0.6000    0.9000
    0.8000    0.3000
```

```
BS = B * S
```

```
BS = 2×5
         0    0.6000    1.5000    0.9000         0
         0    0.8000    1.1000    0.3000         0
```

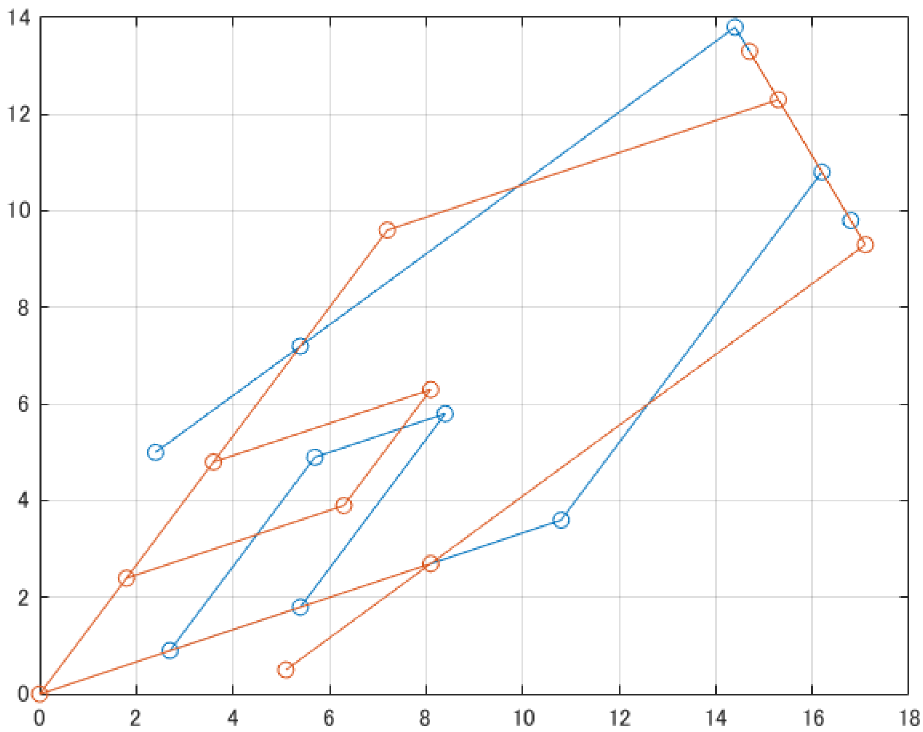
```
plot(AS(1,:), AS(2,:),"o-")
hold on
plot(BS(1,:), BS(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
BH = B * H
```

```
BH = 2×12

         0    8.1000    5.1000   17.1000   14.7000   15.3000    7.2000    1.8000    6.3000
         0    2.7000    0.5000    9.3000   13.3000   12.3000    9.6000    2.4000    3.9000
```

```
plot(AH(1,:), AH(2,:),"o-")
hold on
plot(BH(1,:), BH(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
det(B)
```

```
ans = -0.5400
```

## 性質2: 単位行列

単位行列を掛けても図形は不変なので行列式は1

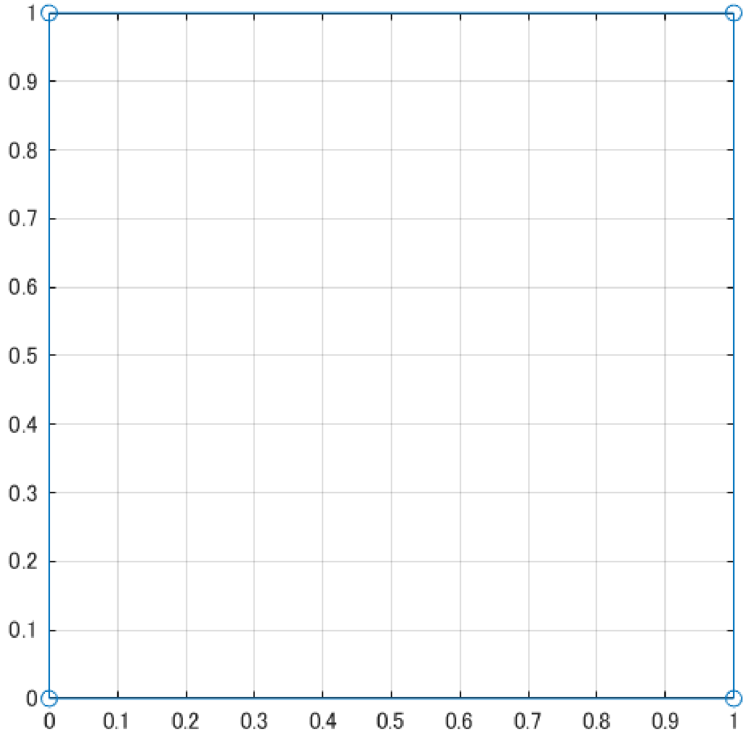
```
B = [1 0; 0 1]
```

```
B = 2×2
     1     0
     0     1
```

```
BS = B * S
```

```
BS = 2×5
     0     1     1     0     0
     0     0     1     1     0
```

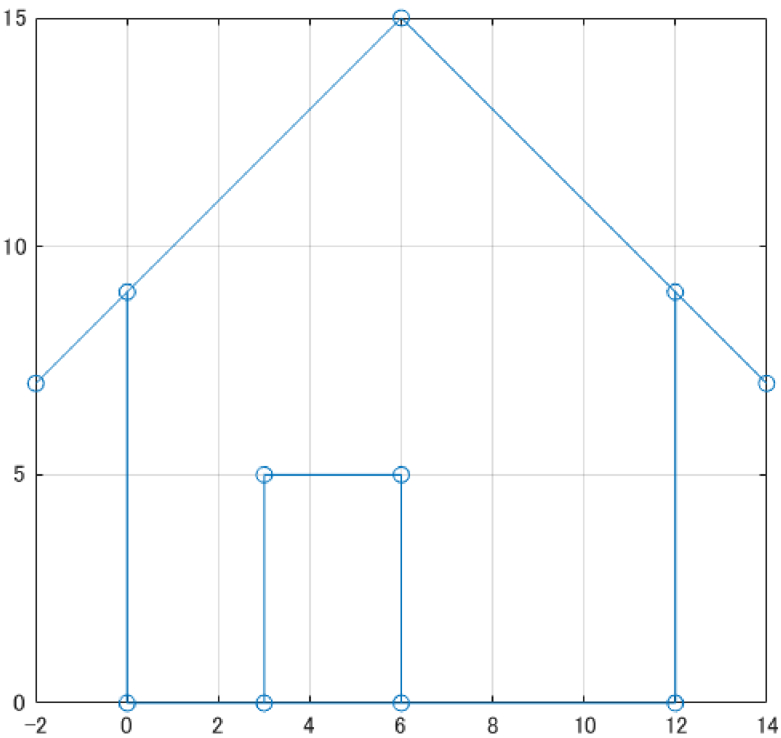
```
plot(BS(1,:), BS(2,:),"o-")
daspect([1 1 1])
grid on
```



```
BH = B * H
```

```
BH = 2×12
     0     0    -2     6    14    12    12     3     3     6     6     0
     0     9     7    15     7     9     0     0     5     5     0     0
```

```
plot(BH(1,:), BH(2,:),"o-")
daspect([1 1 1])
grid on
```




```
det(B)
```

```
ans = 1
```

## 性質3: 多重線形性1

1列目を2倍する. 第1のベクトルに沿って2倍に引き伸ばされる

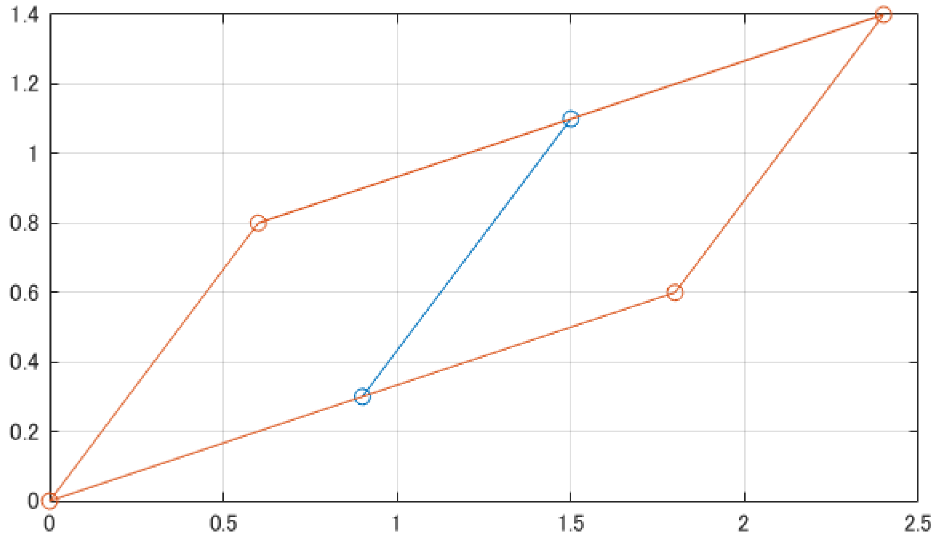
```
B = [1.8 0.6; 0.6 0.8]
```

```
B = 2×2
    1.8000    0.6000
    0.6000    0.8000
```

```
BS = B * S
```

```
BS = 2×5
         0    1.8000    2.4000    0.6000         0
         0    0.6000    1.4000    0.8000         0
```

```
plot(AS(1,:), AS(2,:),"o-")
hold on
plot(BS(1,:), BS(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```

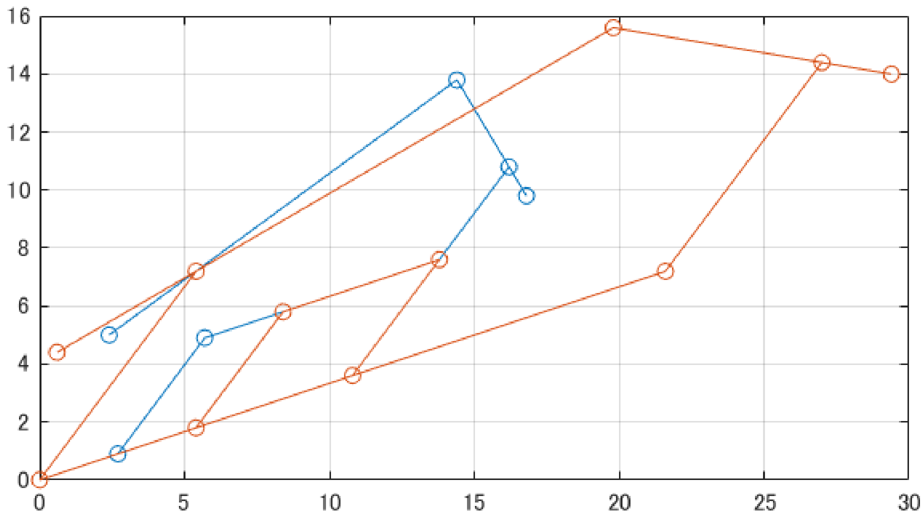



```
BH = B * H
```

```
BH = 2×12
         0    5.4000    0.6000   19.8000   29.4000   27.0000   21.6000    5.4000    8.4000
         0    7.2000    4.4000   15.6000   14.0000   14.4000    7.2000    1.8000    5.8000
```

```
plot(AH(1,:), AH(2,:),"o-")
hold on
plot(BH(1,:), BH(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
det(B)
```

```
ans = 1.0800
```

## 性質3: 多重線形性1

1行目を2倍する. x軸に沿って2倍に引き伸ばされる

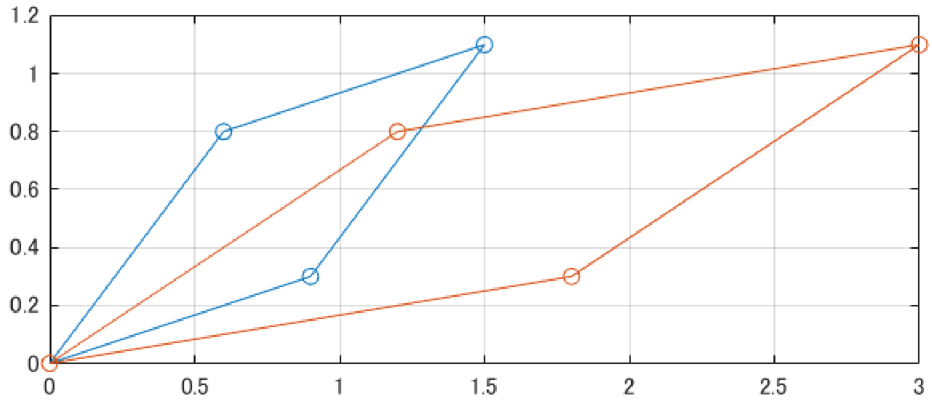
```
B = [1.8 1.2; 0.3 0.8]
```

```
B = 2x2
    1.8000    1.2000
    0.3000    0.8000
```

```
BS = B * S
```

```
BS = 2x5
         0    1.8000    3.0000    1.2000         0
         0    0.3000    1.1000    0.8000         0
```

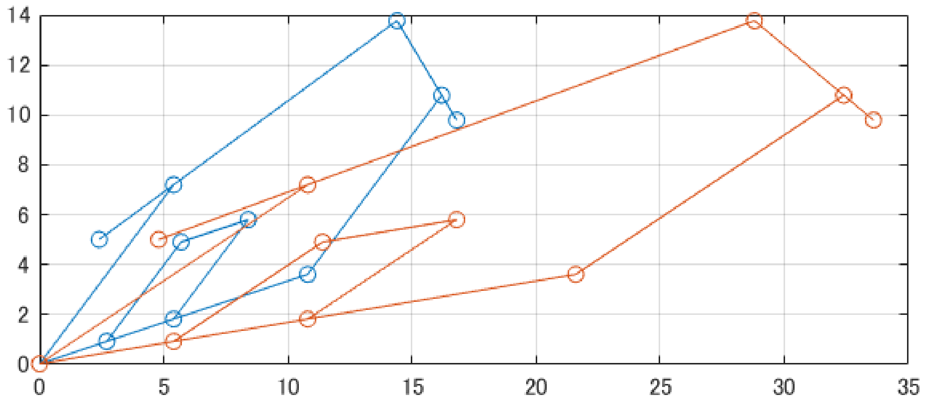
```
plot(AS(1,:), AS(2,:),"o-")
hold on
plot(BS(1,:), BS(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
BH = B * H
```

```
BH = 2×12
          0   10.8000    4.8000   28.8000   33.6000   32.4000   21.6000    5.4000   11.4000
          0    7.2000    5.0000   13.8000    9.8000   10.8000    3.6000    0.9000    4.9000
```

```
plot(AH(1,:), AH(2,:),"o-")
hold on
plot(BH(1,:), BH(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
det(B)
```

```
ans = 1.0800
```

## 性質4: 多重線形性2

ある行が和になっている場合, 行列式は2つの行列式の和

```
A = [ 0.9 0.6; 0.3 0.8]
```

```
A = 2×2
    0.9000    0.6000
    0.3000    0.8000
```

```
B = [ 0.9 0.6; 0.1 0.1]
```

```
B = 2×2
    0.9000    0.6000
    0.1000    0.1000
```

```
C = [ 0.9 0.6; 0.4 0.9]
```

```
C = 2×2
    0.9000    0.6000
    0.4000    0.9000
```

```
AS = A * S
```

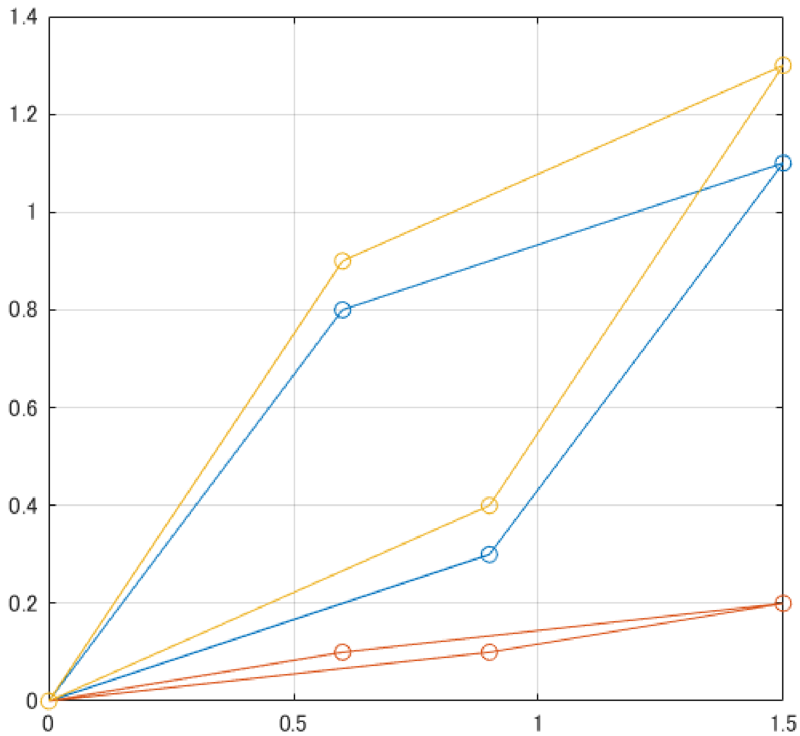
```
AS = 2×5
         0    0.9000    1.5000    0.6000         0
         0    0.3000    1.1000    0.8000         0
```

```
BS = 2×5

         0    0.9000    1.5000    0.6000         0
         0    0.1000    0.2000    0.1000         0
```

```
CS = C * S
```

```
CS = 2×5

         0    0.9000    1.5000    0.6000         0
         0    0.4000    1.3000    0.9000         0
```

```
plot(AS(1,:), AS(2,:),"o-")
hold on
plot(BS(1,:), BS(2,:),"o-")
plot(CS(1,:), CS(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
AH = A * H
```

```
AH = 2×12

         0    5.4000    2.4000   14.4000   16.8000   16.2000   10.8000    2.7000    5.7000
         0    7.2000    5.0000   13.8000    9.8000   10.8000    3.6000    0.9000    4.9000
```

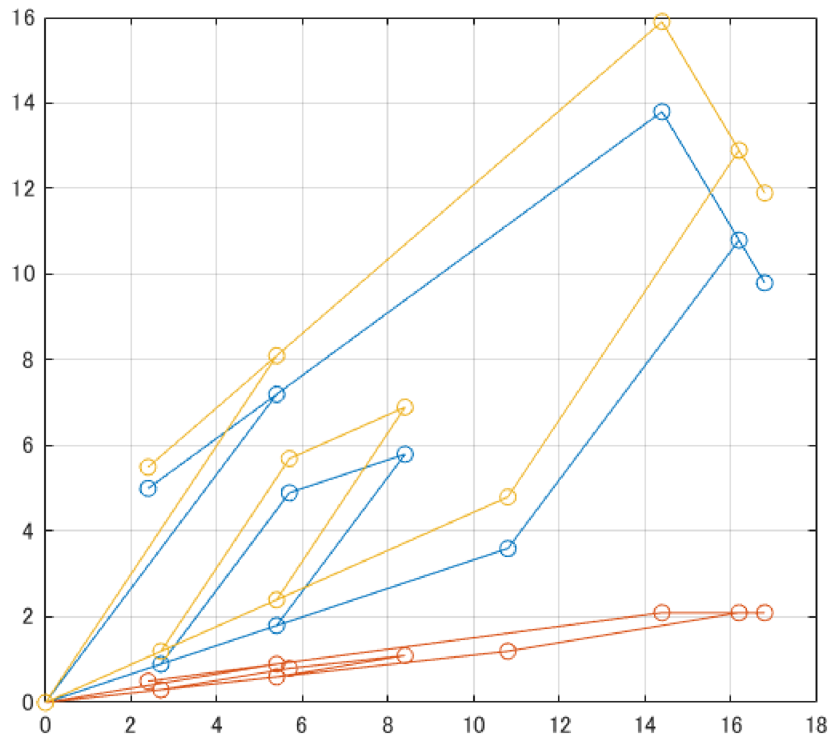
```
BH = B * H
```

```
BH = 2×12

         0    5.4000    2.4000   14.4000   16.8000   16.2000   10.8000    2.7000    5.7000
         0    0.9000    0.5000    2.1000    2.1000    2.1000    1.2000    0.3000    0.8000
```

```
CH = C * H
```

```
CH = 2×12
         0    5.4000    2.4000   14.4000   16.8000   16.2000   10.8000    2.7000    5.7000
         0    8.1000    5.5000   15.9000   11.9000   12.9000    4.8000    1.2000    5.7000
```

```
plot(AH(1,:), AH(2,:),"o-")
hold on
plot(BH(1,:), BH(2,:),"o-")
plot(CH(1,:), CH(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
det(A)
```

```
ans = 0.5400
```

```
det(B)
```

```
ans = 0.0300
```

```
det(C)
```

```
ans = 0.5700
```

## 性質4: 多重線形性2

ある列が和になっている場合, 行列式は2つの行列式の和

```
A = [ 0.9 0.6; 0.3 0.8]
```

```
A = 2×2
    0.9000    0.6000
    0.3000    0.8000
```

```
B = 2×2
    0.2000    0.6000
    0.1000    0.8000
```

```
C = [ 1.1 0.6; 0.4 0.8]
```

```
C = 2×2
    1.1000    0.6000
    0.4000    0.8000
```

```
AS = A * S
```

```
AS = 2×5
         0    0.9000    1.5000    0.6000         0
         0    0.3000    1.1000    0.8000         0
```

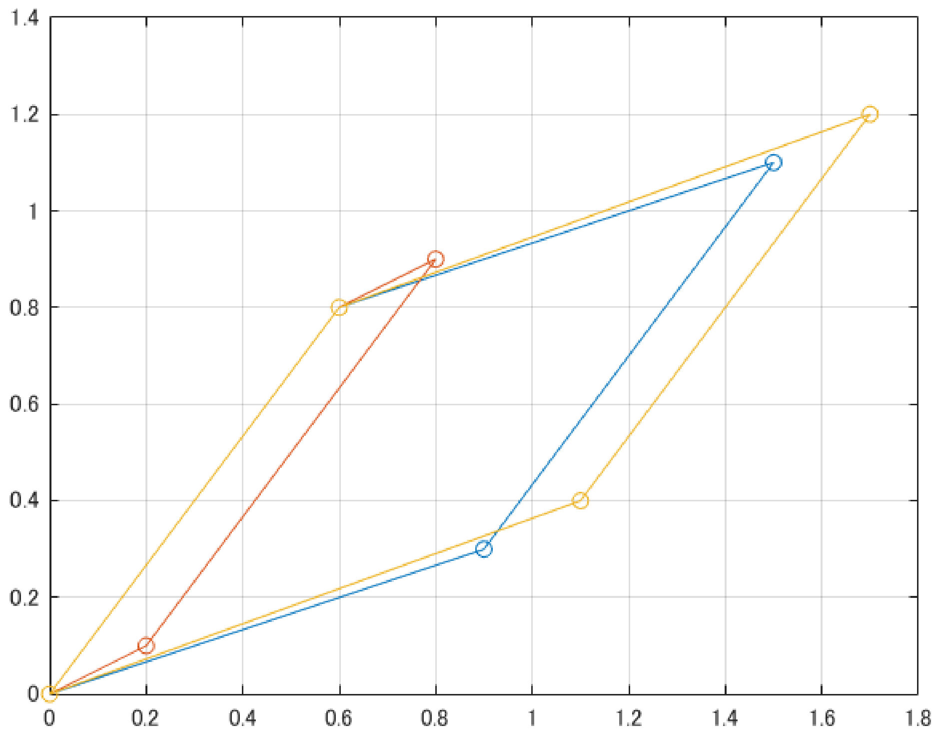
```
BS = B * S
```

```
BS = 2×5
         0    0.2000    0.8000    0.6000         0
         0    0.1000    0.9000    0.8000         0
```

```
CS = C * S
```

```
CS = 2×5
         0    1.1000    1.7000    0.6000         0
         0    0.4000    1.2000    0.8000         0
```

```
plot(AS(1,:), AS(2,:),"o-")
hold on
plot(BS(1,:), BS(2,:),"o-")
plot(CS(1,:), CS(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
AH = A * H
```

```
AH = 2×12
         0    5.4000    2.4000   14.4000   16.8000   16.2000   10.8000    2.7000    5.7000
         0    7.2000    5.0000   13.8000    9.8000   10.8000    3.6000    0.9000    4.9000
```

```
BH = B * H
```

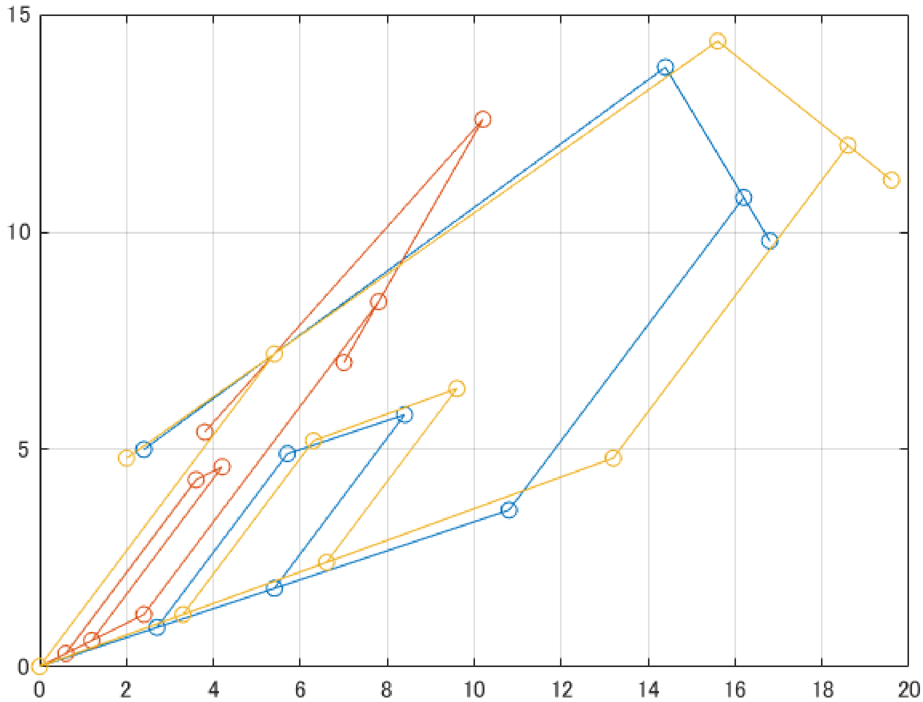
```
BH = 2×12
         0    5.4000    3.8000   10.2000    7.0000    7.8000    2.4000    0.6000    3.6000
         0    7.2000    5.4000   12.6000    7.0000    8.4000    1.2000    0.3000    4.3000
```

```
CH = C * H
```

```
CH = 2×12
         0    5.4000    2.0000   15.6000   19.6000   18.6000   13.2000    3.3000    6.3000
         0    7.2000    4.8000   14.4000   11.2000   12.0000    4.8000    1.2000    5.2000
```

```
plot(AH(1,:), AH(2,:),"o-")
hold on
plot(BH(1,:), BH(2,:),"o-")
plot(CH(1,:), CH(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
det(A)
```

```
ans = 0.5400
```

```
det(B)
```

```
ans = 0.1000
```

```
det(C)
```

```
ans = 0.6400
```

## 性質5: 2つの行が等しければ行列式は0

変換先のベクトルが平行となるため, 図形は潰れて面積0

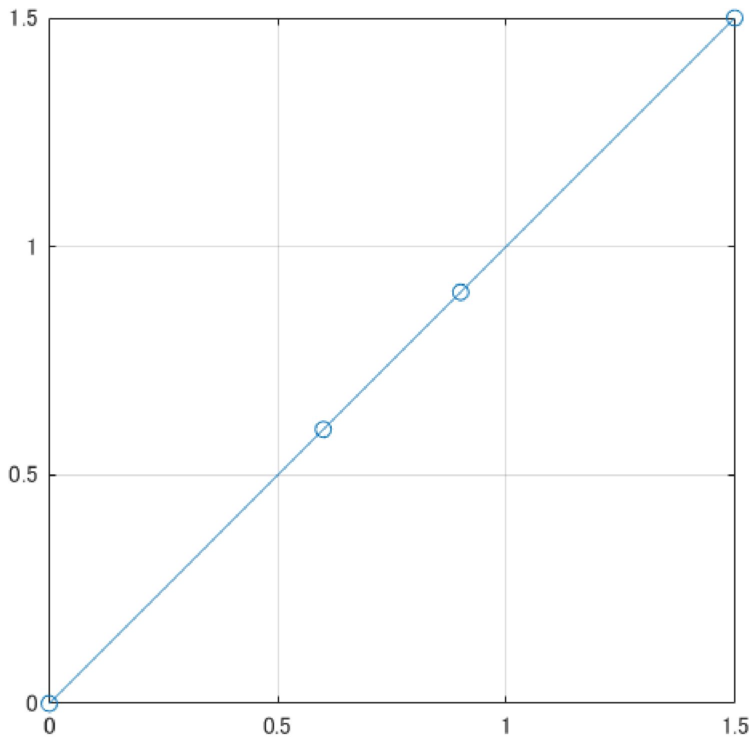
```
B = [0.9 0.6; 0.9 0.6]
```

```
B = 2×2
    0.9000    0.6000
    0.9000    0.6000
```

```
BS = B * S
```

```
BS = 2×5
         0    0.9000    1.5000    0.6000         0
         0    0.9000    1.5000    0.6000         0
```

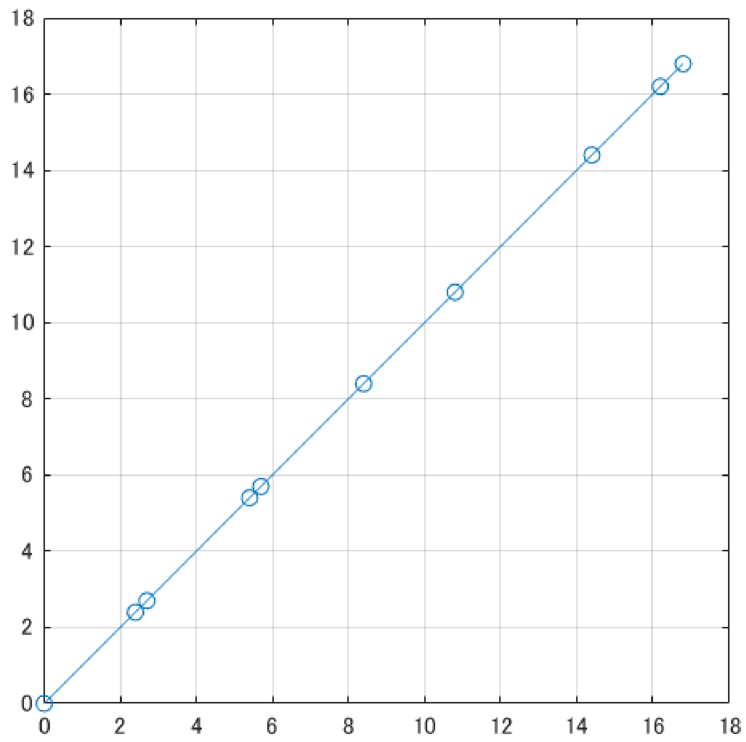
```
plot(BS(1,:), BS(2,:),"o-")
daspect([1 1 1])
grid on
```




```
BH = B * H
```

```
BH = 2×12
         0    5.4000    2.4000   14.4000   16.8000   16.2000   10.8000    2.7000    5.7000
         0    5.4000    2.4000   14.4000   16.8000   16.2000   10.8000    2.7000    5.7000
```

```
plot(BH(1,:), BH(2,:),"o-")
daspect([1 1 1])
grid on
```




```
det(B)
```

```
ans = 0
```

## 性質5: 2つの列が等しければ行列式は0

変換先のベクトルが同一となるため, 図形は潰れて面積0

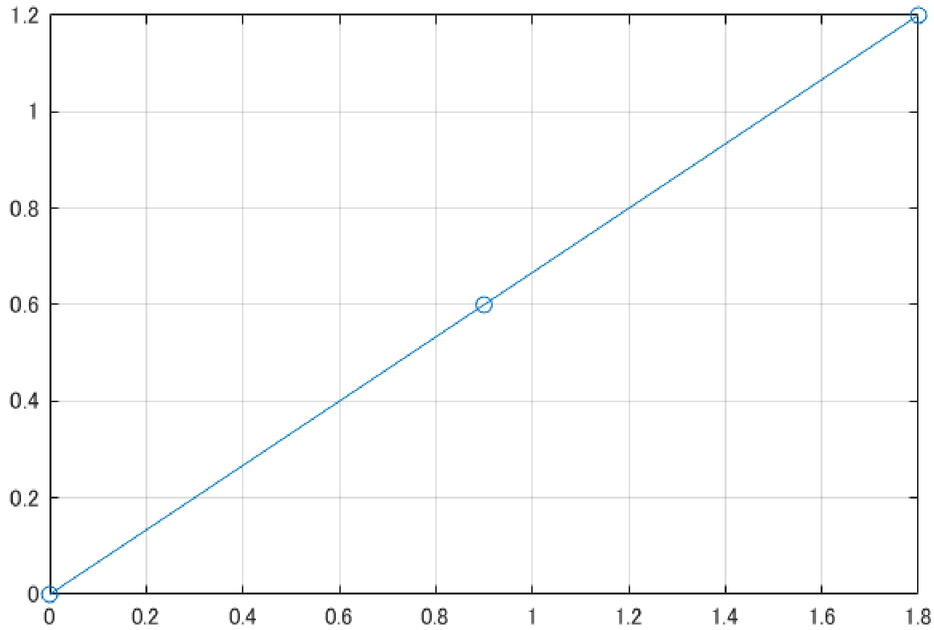
```
B = [0.9 0.9; 0.6 0.6]
```

```
B = 2×2
    0.9000    0.9000
    0.6000    0.6000
```

```
BS = B * S
```

```
BS = 2×5
         0    0.9000    1.8000    0.9000         0
         0    0.6000    1.2000    0.6000         0
```

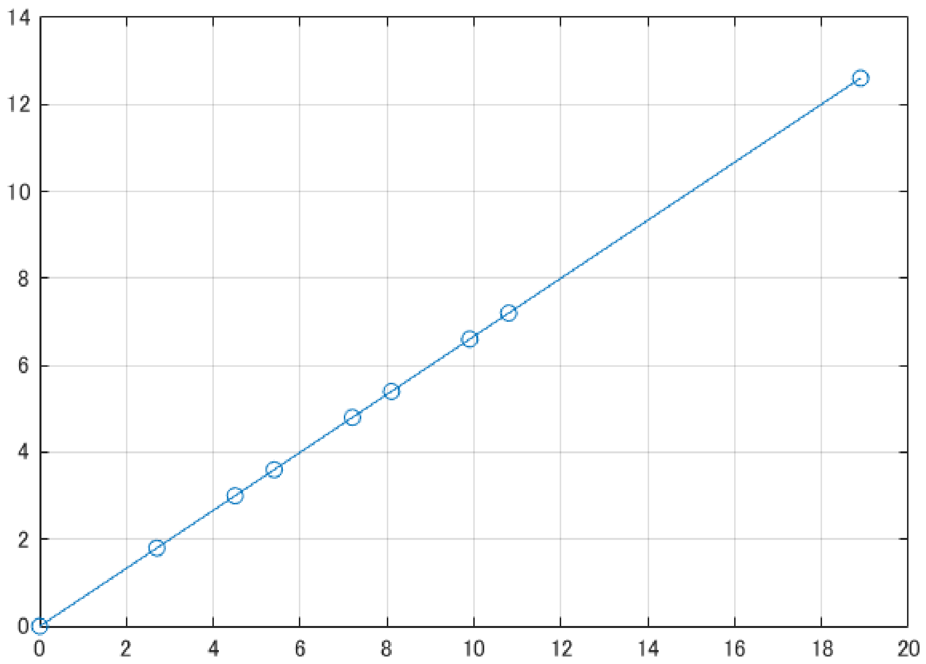
```
plot(BS(1,:), BS(2,:),"o-")
daspect([1 1 1])
grid on
```




```
BH = B * H
```

```
BH = 2×12
         0    8.1000    4.5000   18.9000   18.9000   18.9000   10.8000    2.7000    7.2000
         0    5.4000    3.0000   12.6000   12.6000   12.6000    7.2000    1.8000    4.8000
```

```
plot(BH(1,:), BH(2,:),"o-")
daspect([1 1 1])
grid on
```




```
det(B)
```

```
ans = -3.3307e-18
```

## 性質6: ある行の倍数を別の行から引いても, その行列式の値は変わらない

1行目に2行目の0.5倍を加える. 平行四辺形の面積は不変

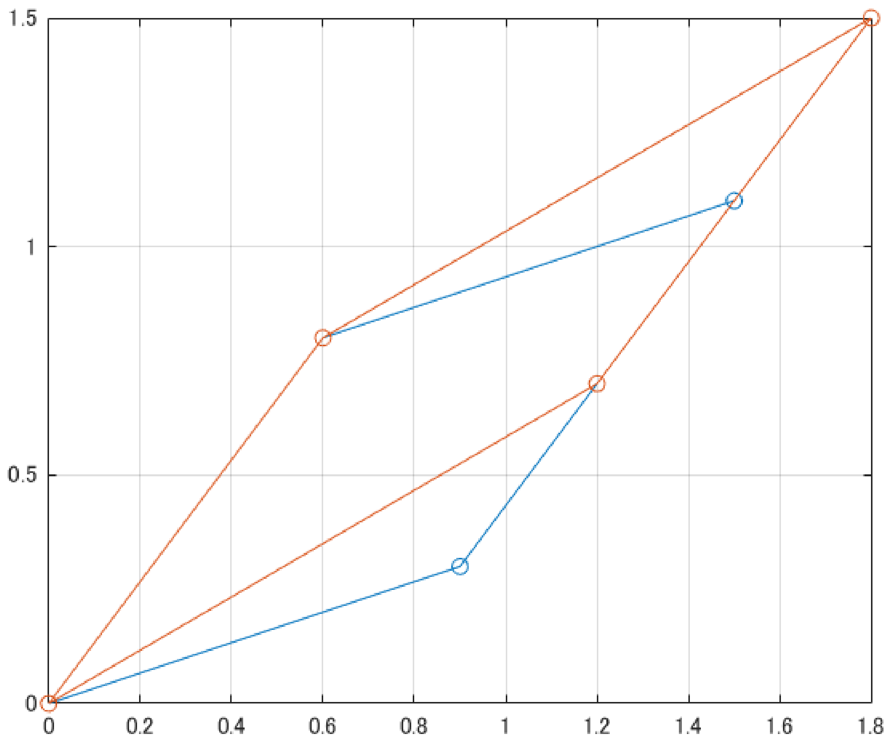
```
B = [1.2 0.6; 0.7 0.8]
```

```
B = 2×2
    1.2000    0.6000
    0.7000    0.8000
```

```
BS = B * S
```

```
BS = 2×5
         0    1.2000    1.8000    0.6000         0
         0    0.7000    1.5000    0.8000         0
```

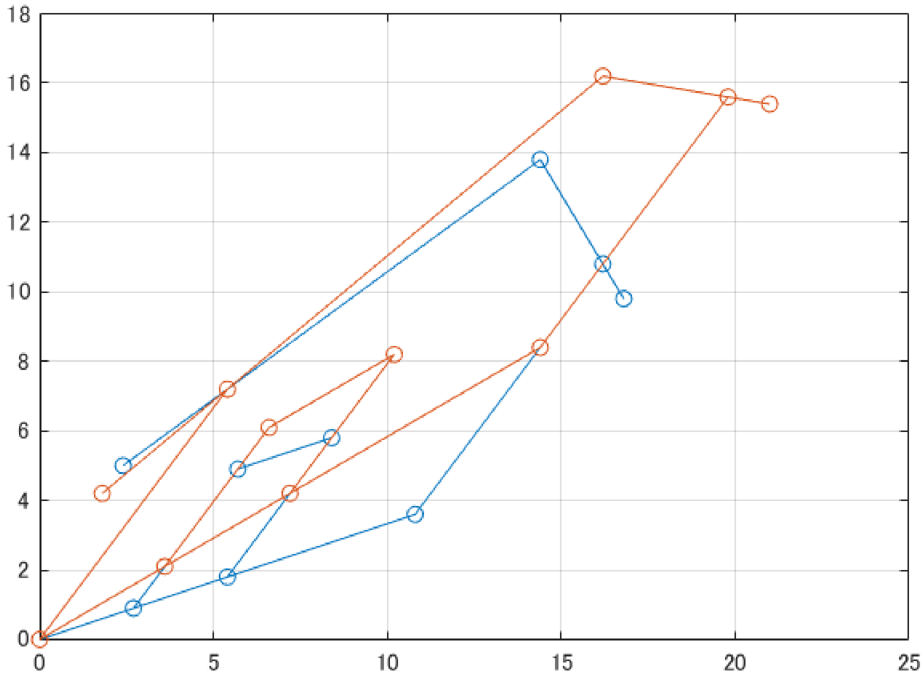
```
plot(AS(1,:), AS(2,:),"o-")
hold on
plot(BS(1,:), BS(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
BH = B * H
```

```
BH = 2×12

         0    5.4000    1.8000   16.2000   21.0000   19.8000   14.4000    3.6000    6.6000
         0    7.2000    4.2000   16.2000   15.4000   15.6000    8.4000    2.1000    6.1000
```

```
plot(AH(1,:), AH(2,:),"o-")
hold on
plot(BH(1,:), BH(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
det(B)
```

```
ans = 0.5400
```

## 性質6: ある列の倍数を別の列から引いても, その行列式の値は変わらない

2列目に1列目の1/3倍を加える. x軸の値に比例してy軸方向にずれる. 平行四辺形の面積は不変

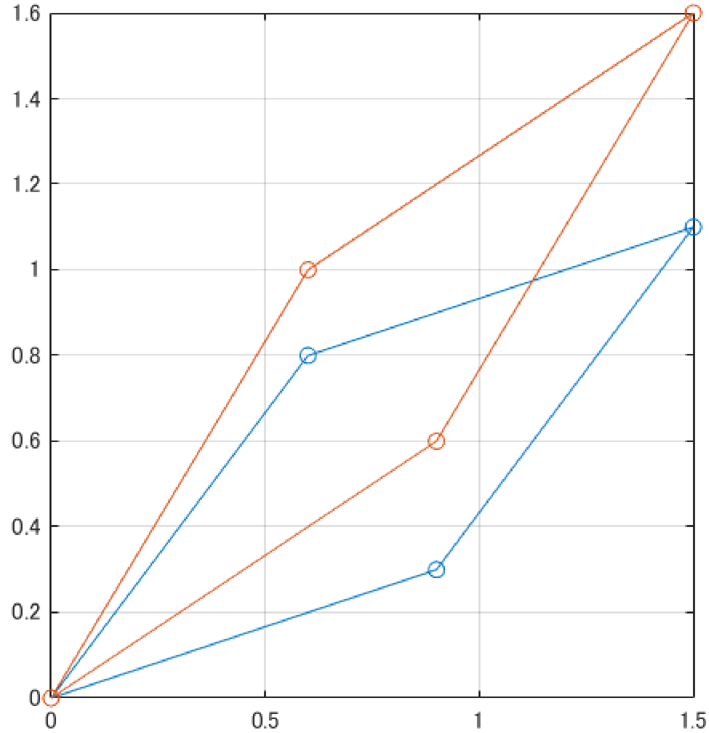
```
B = [0.9 0.6; 0.6 1.0]
```

```
B = 2×2
    0.9000    0.6000
    0.6000    1.0000
```

```
BS = B * S
```

```
BS = 2×5
         0    0.9000    1.5000    0.6000         0
         0    0.6000    1.6000    1.0000         0
```

```
plot(AS(1,:), AS(2,:),"o-")
hold on
plot(BS(1,:), BS(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```

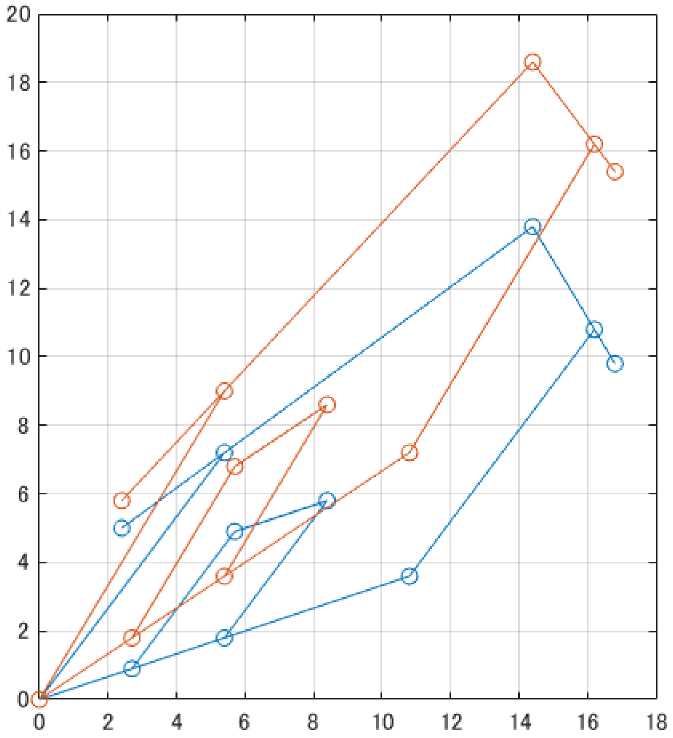



```
BH = B * H
```

BH = 2×12

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 5.4000 | 2.4000 | 14.4000 | 16.8000 | 16.2000 | 10.8000 | 2.7000 | 5.7000 |
| 0 | 9.0000 | 5.8000 | 18.6000 | 15.4000 | 16.2000 | 7.2000 | 1.8000 | 6.8000 |

```
plot(AH(1,:), AH(2,:),"o-")
hold on
plot(BH(1,:), BH(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
det(B)
```

```
ans = 0.5400
```

# 性質7．三角行列の行列式は，その3角行列の対角成分の積に等しい．

(1,2)成分の値を変えても, 平行四辺形の面積は変化しない

```
A = [0.9 0.6; 0.0 1.0]
```

```
A = 2×2
    0.9000    0.6000
         0    1.0000
```

```
B = [0.9 1.0; 0.0 1.0]
```

```
B = 2×2
    0.9000    1.0000
         0    1.0000
```

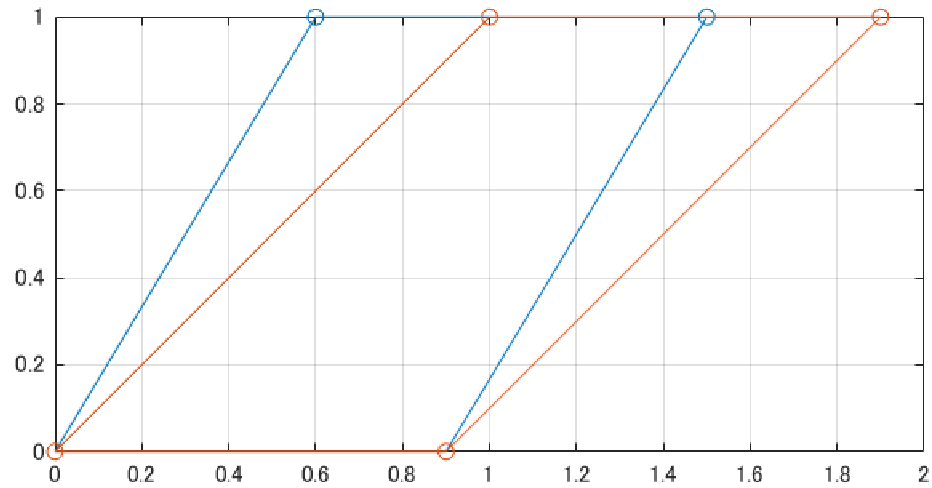
```
AS = A * S
```

```
AS = 2×5
         0    0.9000    1.5000    0.6000         0
         0         0    1.0000    1.0000         0
```

```
BS = B * S
```

```
BS = 2×5

         0    0.9000    1.9000    1.0000         0
         0         0    1.0000    1.0000         0
```

```
plot(AS(1,:), AS(2,:),"o-")
hold on
plot(BS(1,:), BS(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```



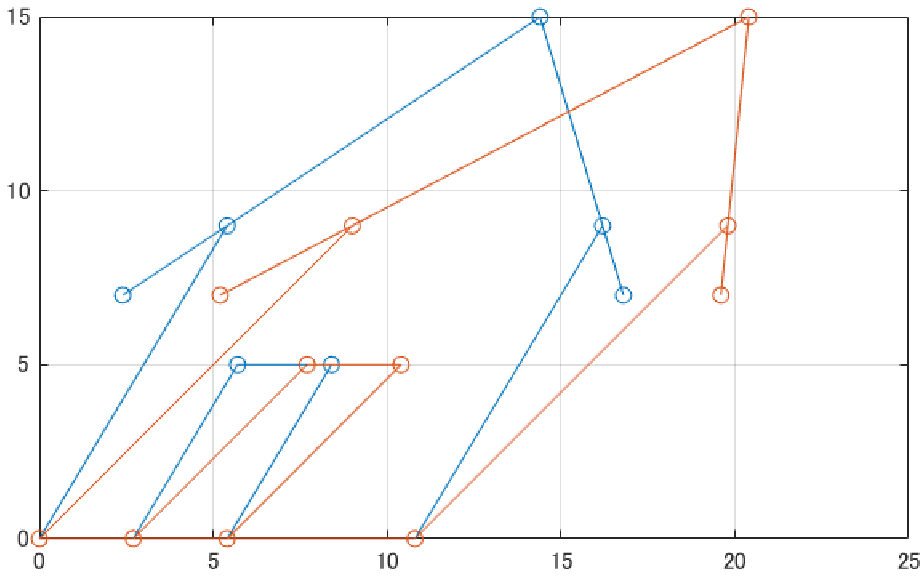

```
AH = A * H
```

```
AH = 2×12

         0    5.4000    2.4000   14.4000   16.8000   16.2000   10.8000    2.7000    5.7000
         0    9.0000    7.0000   15.0000    7.0000    9.0000         0         0    5.0000
```

```
BH = B * H
```

```
BH = 2×12

         0    9.0000    5.2000   20.4000   19.6000   19.8000   10.8000    2.7000    7.7000
         0    9.0000    7.0000   15.0000    7.0000    9.0000         0         0    5.0000
```

```
plot(AH(1,:), AH(2,:),"o-")
hold on
plot(BH(1,:), BH(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
det(A)
```

```
ans = 0.9000
```

```
det(B)
```

```
ans = 0.9000
```

## 性質8. 転置行列の行列式は元の行列の行列式に等しい.

```
A = [0.9 0.6; 0.2 1.0]
```

```
A = 2×2
    0.9000    0.6000
    0.2000    1.0000
```

```
B = transpose(A)
```

```
B = 2×2
    0.9000    0.2000
    0.6000    1.0000
```

```
AS = A * S
```

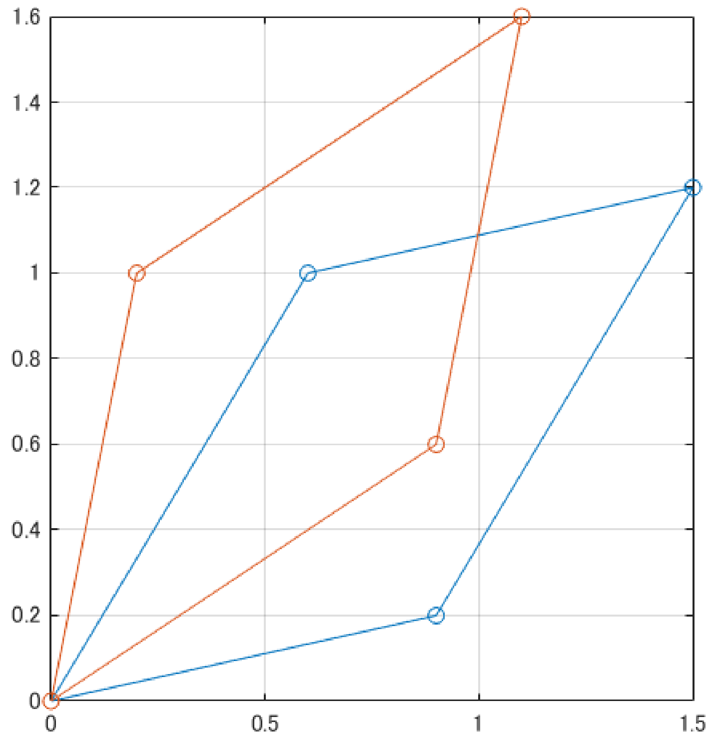
```
AS = 2×5
         0    0.9000    1.5000    0.6000         0
         0    0.2000    1.2000    1.0000         0
```

```
BS = B * S
```

```
BS = 2×5
```

```
         0    0.9000    1.1000    0.2000         0
         0    0.6000    1.6000    1.0000         0
```

```
plot(AS(1,:), AS(2,:),"o-")
hold on
plot(BS(1,:), BS(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```



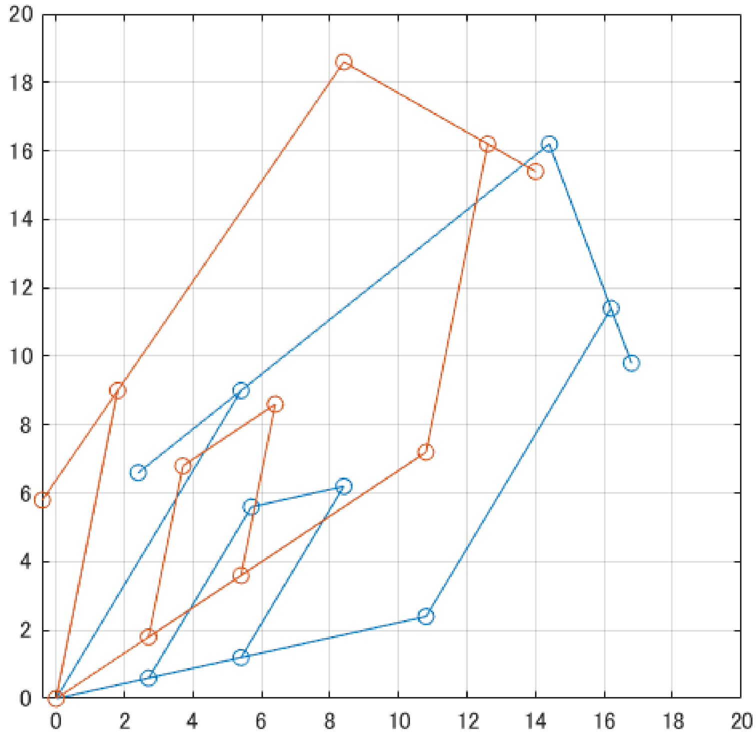
```
AH = A * H
```

```
AH = 2×12
         0    5.4000    2.4000   14.4000   16.8000   16.2000   10.8000    2.7000    5.7000
         0    9.0000    6.6000   16.2000    9.8000   11.4000    2.4000    0.6000    5.6000
```

```
BH = B * H
```

```
BH = 2×12
         0    1.8000   -0.4000    8.4000   14.0000   12.6000   10.8000    2.7000    3.7000
         0    9.0000    5.8000   18.6000   15.4000   16.2000    7.2000    1.8000    6.8000
```

```
plot(AH(1,:), AH(2,:),"o-")
hold on
plot(BH(1,:), BH(2,:),"o-")
hold off
daspect([1 1 1])
grid on
```




```
det(A)
```

```
ans = 0.7800
```

```
det(B)
```

```
ans = 0.7800
```

## 性質9. $AB$ の行列式はそれぞれの行列式の積に等しい

図形に $AB$ を掛けると, まず $|B|$ 倍されさらに $|A|$ 倍される

## 例題

次の行列の行列式を求め, MATLABで答え合わせを行え

1. $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
2. $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$