クレジット:
Mathematics and Informatics Center 文科系のための線形代数・解析Ⅱ
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一般化 $f(g_1, g_2 \cdots g_m)$, $g_i(h_1, h_2 \cdots h_n)$

$$h_j(x_1 \cdots x_l)$$

$$\begin{array}{cc} & l \\ 1 & \left[\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \cdots \dfrac{\partial f}{\partial x_l} \right] \end{array} = \begin{array}{cc} & m \\ 1 & \left[f_{g_1}, \cdots f_{g_m} \right] \end{array} \begin{array}{cc} & n \\ m & \boxed{\dfrac{\partial g_i}{\partial h_j}} \end{array}$$

$$\frac{\partial f}{\partial g_1} \cdots \frac{\partial f}{\partial g_m}$$

$$\times \begin{array}{cc} & l \\ n & \boxed{\dfrac{\partial h_j}{\partial x_k}} \end{array}$$

極座標　例題

$$z = x^2 + xy + y^2 \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\frac{\partial z}{\partial x} = 2x + y$$

$$\frac{\partial z}{\partial y} = x + 2y$$

$$\begin{bmatrix} \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix}$$

$$= \begin{bmatrix} 2x+y, & x+2y \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= \left[(2x+y)\cos\theta + (x+2y)\sin\theta, \right.$$
$$\left. -(2x+y)r\sin\theta + (x+2y)r\cos\theta \right]$$

$$\frac{\partial z}{\partial r} = 2r\cos^2\theta + r\sin\theta\cos\theta + r\sin\theta\cos\theta + 2r\sin^2\theta$$
$$= 2r + 2r\sin\theta\cos\theta = r(2+\sin 2\theta)$$

$$\frac{\partial z}{\partial \theta} = r^2\cos 2\theta$$

$z = z(x, y)$ $\quad \dfrac{\partial^2 z}{\partial r^2}, \dfrac{\partial^2 z}{\partial \theta^2}$ を求めよ

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\sin\theta$$

$$= \frac{\partial}{\partial x}\underline{\underline{\frac{\partial z}{\partial r}}}\cos\theta + \frac{\partial}{\partial y}\underline{\underline{\frac{\partial z}{\partial r}}}\sin\theta$$

$$= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 z}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta$$

$$\boxed{\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\underline{\frac{1}{r}\frac{\partial z}{\partial r}}$$

$$\boxed{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}}=\underline{\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{1}{r}\frac{\partial}{\partial r}}$$

演算子 operator

ラプラシアン

Δ

∇^2

Δz , $\nabla^2 z$

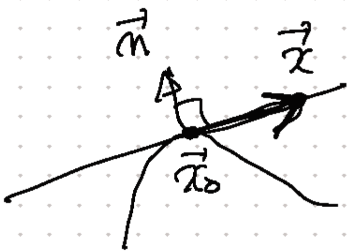
接線の方程式

$$y = f(x_0) + f'(x_0)(x - x_0)$$

テーラー展開の一次

$$f'(x_0)(x - x_0) - (y - f(x_0)) = 0$$

$$\underbrace{[f'(x_0), -1]}_{\vec{n}} \cdot \underbrace{[x - x_0, y - f(x_0)]}_{\vec{x} - \vec{x_0}} = 0$$



$$\boxed{\vec{n} \cdot (\vec{x} - \vec{x_0}) = 0}$$

$$f(x,y) \simeq f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

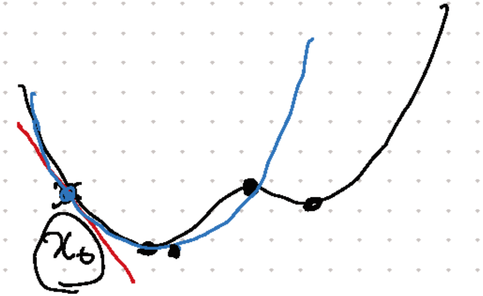
$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (f - f(x_0,y_0)) = 0$$

$$\underbrace{[\, f_x(x_0,y_0),\ f_y(x_0,y_0),\ -1 \,]}_{\vec{n}} \cdot [\, \vec{x} - \vec{x}_0 \,] = 0$$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \\ f(x_0,y_0) \end{bmatrix}$$

ニュートン法　(一次元)

$$y = f(x)$$



$x = x_0$ のまわりで　テーラー展開

$$y \fallingdotseq f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

$$\frac{dy}{dx} = f'(x_0) + f''(x_0)(x - x_0) = 0$$

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)}$$

$$x_0 \to x_1 \to x_2 \to x_3 \to \cdots$$

$z = f(x, y)$

(x_0, y_0) のまわりで テイラー展開

$$f(x,y) = f(x_0, y_0) + f_x (x - x_0) + f_y (y - y_0)$$
$$+ \frac{1}{2} f_{xx} (x - x_0)^2 + f_{xy} (x - x_0)(y - y_0)$$
$$+ \frac{1}{2} f_{yy} (y - y_0)^2$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$

$$\begin{cases} f_x + f_{xx}(x - x_0) + f_{xy}(y - y_0) = 0 \\ f_y + f_{xy}(x - x_0) + f_{yy}(y - y_0) = 0 \end{cases}$$

ヘッセ行列

$$\nabla f(\vec{x}_0) + H(\vec{x} - \vec{x}_0) = 0$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}_{x_0 y_0} + \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = 0$$

$$\vec{x}_1 = \vec{x}_0 - H^{-1} \nabla f(\vec{x}_0).$$

$$\vec{x}_0 \longrightarrow \vec{x}_1 \longrightarrow \vec{x}_2 \longrightarrow \vec{x}_3 \cdots\cdots$$

勾配降下法

$$\nabla f(x_0)$$

$$\vec{x} = \vec{x}_0 - \varepsilon \nabla f(x_0) \qquad \varepsilon \ll 1$$

$$\begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix}_{x_0 y_0}$$

$$f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$$

$$f_x = y - \frac{1}{x^2}$$

$$f_y = x - \frac{1}{y^2}$$

$(x, y) = (2, 3)$

$f_x(2, 3) = 3 - \frac{1}{4} = \frac{11}{4}$

$f_y(2, 3) = 2 - \frac{1}{9} = \frac{17}{9}$

∇f

$\varepsilon = 0.1$

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 0.1 \begin{bmatrix} \frac{11}{4} \\ \frac{17}{9} \end{bmatrix}$$

極小点　$f_x = f_y = 0$

$y - \frac{1}{x^2} = 0$

$x - \frac{1}{y^2} = 0 \quad \rightarrow \quad x - x^4 = 0$

$x(1 - x^3) = 0$

$\rightarrow \boxed{x = 1} \quad \boxed{y = 1}$

三次元の場合　$F(x,y,z)=0$　曲面

$$\longrightarrow \quad z=f(x,y)$$

勾配

$$\nabla F = \begin{bmatrix} \dfrac{\partial F}{\partial x} \\ \dfrac{\partial F}{\partial y} \\ \dfrac{\partial F}{\partial z} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

微分　$F(x,y,f(x,y))=0$　を x でびぶん

$$F_x + F_z \frac{\partial f}{\partial x} = 0$$

$$F_y + F_z \frac{\partial f}{\partial y} = 0$$

$$\nabla F = \begin{bmatrix} -F_z \dfrac{\partial f}{\partial x} \\ -F_z \dfrac{\partial f}{\partial y} \\ F_z \end{bmatrix} = -F_z \underbrace{\begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ -1 \end{bmatrix}}_{\vec{n}}$$

$$\nabla F \parallel \vec{n}$$