クレジット：
Mathematics and Informatics Center 文科系のための線形代数・解析Ⅱ
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合成関数の微分

一変数関数の場合 $y = f(g(x))$ $\dfrac{dy}{dx} = \dfrac{df}{dg} \cdot \dfrac{dg}{dx}$

二変数関数 $y = f(g_1(x_1, x_2), g_2(x_1, x_2))$

$$\frac{\partial y}{\partial x_1} = \frac{\partial f}{\partial g_1} \cdot \frac{\partial g_1}{\partial x_1} + \frac{\partial f}{\partial g_2} \cdot \frac{\partial g_2}{\partial x_1}$$

$$\frac{\partial y}{\partial x_2} = \frac{\partial f}{\partial g_1} \cdot \frac{\partial g_1}{\partial x_2} + \frac{\partial f}{\partial g_2} \cdot \frac{\partial g_2}{\partial x_2}$$

$\because$ $f(g_1(x_1 + \Delta x_1, x_2), g_2(x_1 + \Delta x_1, x_2))$

$$= f\left(g_1(x_1, x_2) + \frac{\partial g_1}{\partial x_1} \Delta x_1, \ g_2(x_1, x_2) + \frac{\partial g_2}{\partial x_1} \Delta x_1\right) \quad (+\text{高次})$$

$$= f(g_1(x_1, x_2), g_2(x_1, x_2)) + \frac{\partial f}{\partial g_1} \cdot \frac{\partial g_1}{\partial x_1} \Delta x_1 + \frac{\partial f}{\partial g_2} \cdot \frac{\partial g_2}{\partial x_1} \Delta x_1$$

$$\frac{\partial f}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{\underbrace{f(g_1(x_1 + \Delta x_1, x_2), g_2(x_1 + \Delta x_1, x_2)) - f(g_1, g_2)}}{\Delta x_1}$$

$$= \frac{\partial f}{\partial g_1} \cdot \frac{\partial g_1}{\partial x_1} + \frac{\partial f}{\partial g_2} \cdot \frac{\partial g_2}{\partial x_1}$$

一般変数の場合

$$y = f(g_1, \cdots, g_m) \qquad g_i = g_i(x_1, \cdots, x_n)$$

$$\boxed{\frac{\partial f}{\partial x_j} = \sum_{i=1}^{m} \frac{\partial f}{\partial g_i} \cdot \frac{\partial g_i}{\partial x_j}}$$

行列表記

$$\frac{Df}{Dx} = \left[\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right] \qquad 1 \times n \text{ 行列}$$

$$\frac{Df}{Dg} = \left[\frac{\partial f}{\partial g_1}, \cdots, \frac{\partial f}{\partial g_m} \right] \qquad 1 \times m \text{ 行列}$$

$$\frac{Dg}{Dx} = \left[\frac{\partial g_i}{\partial x_j} \right]_{1 \leq i \leq m,\ 1 \leq j \leq n} \qquad m \times n \text{ 行列}$$

$$\frac{Df}{Dx} = \frac{Df}{Dg} \cdot \frac{Dg}{Dx}$$ 行列の積として書ける.

例)

$$f(x,y) = \log\left(1 - (x^2+y^2-2)^2\right)$$

$$= \log\left(1 - g(x,y)^2\right)$$

$$\left\{\begin{array}{l} n=1 \\ n=2 \end{array}\right.$$

$$g(x,y) = x^2+y^2-2$$

$$\frac{df}{dg} = \frac{-2g}{1-g^2}, \quad \frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = 2y$$

$$\frac{\partial f}{\partial x} = \frac{df}{dg} \cdot \frac{\partial g}{\partial x} = \frac{-4gx}{1-g^2} = \frac{-4x(x^2+y^2-2)}{1-(x^2+y^2-2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{df}{dg} \cdot \frac{\partial g}{\partial y} = \frac{-4y(x^2+y^2-2)}{1-(x^2+y^2-2)^2}$$

Jacobian

- 二変数の場合　$f_1(x_1, x_2)$, $f_2(x_1, x_2)$

Jacobian $\quad \dfrac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = \dfrac{\partial f_1}{\partial x_1} \cdot \dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_1}{\partial x_2} \cdot \dfrac{\partial f_2}{\partial x_1}$

$$\frac{Df}{Dx} = \left[\frac{\partial f_i}{\partial x_j} \right]_{1 \leq i \leq 2,\ 1 \leq j \leq 2}$$

$$\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = \det\left(\frac{Df}{Dx} \right)$$

- n 変数の場合

$f_1(x_1, \cdots, x_n)$, $\cdots$, $f_n(x_1, \cdots, x_n)$

Jacobian $\quad \dfrac{Df}{Dx} = \left[\dfrac{\partial f_i}{\partial x_j} \right]_{1 \leq i \leq n,\ 1 \leq j \leq n}$

$$\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)} = \det\left(\frac{Df}{Dx} \right)$$

性質

・線形性

$$\frac{\partial(f_1, \cdots, a g_i + b h_i, \cdots, f_n)}{\partial(x_1, \cdots, x_n)} = a\frac{\partial(f_1, \cdots, g_i, \cdots, f_n)}{\partial(x_1, \cdots, x_n)} + b\frac{\partial(f_1, \cdots, h_i, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}$$

∘ $\left.\begin{matrix} f_i \leftrightarrow f_j \\ x_i \leftrightarrow x_j \end{matrix}\right)$ などに対して符号がかわる

∘ 合成関数

$$f_i(g_1, \cdots, g_n) \qquad g_i(x_1, \cdots, x_n)$$

$$\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)} = \frac{\partial(f_1, \cdots, f_n)}{\partial(g_1 \cdots g_n)} \cdot \frac{\partial(g_1 \cdots g_n)}{\partial(x_1 \cdots x_n)}$$

☺ $$\frac{Df}{Dx} = \frac{Df}{Dg} \cdot \frac{Dg}{Dx}$$

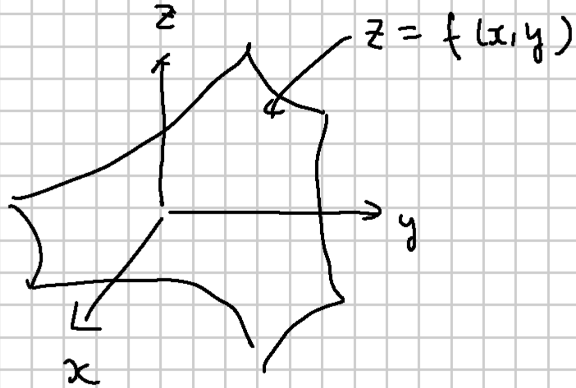
$$\det\left(\frac{Df}{Dx}\right) = \frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}$$

$$\det(AB) = \det(A)\det(B)$$

▷ 等高線に対する法線ベクトルと勾配ベクトル

$$z = f(x, y)$$

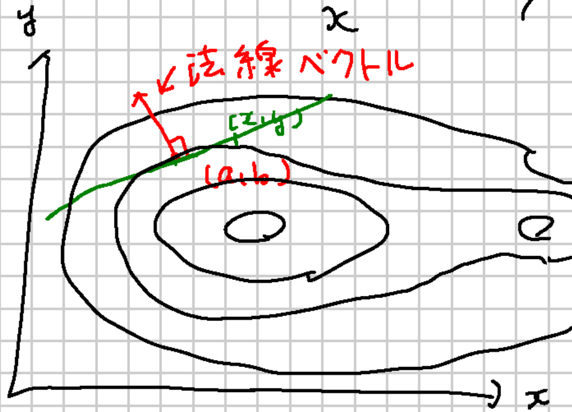
三次元空間内の曲面



$f(x, y) =$ 一定　等高線

$$\cancel{f(a,b)} + \frac{\partial f}{\partial x} \cdot (x-a) + \frac{\partial f}{\partial y}(y-b)$$

$$= \cancel{f(a,b)}$$



$f_x = \dfrac{\partial f}{\partial x}$, $f_y = \dfrac{\partial f}{\partial y}$ … 記号

$$\boxed{f_x (x-a) + f_y (y-b) = 0}$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \vec{x}_0 = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \vec{n} = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

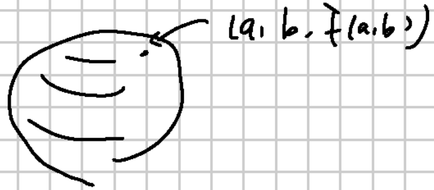
$\vec{n}$: 法線ベクトル.

"ナブラ"

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

勾配ベクトル

接平面と法線ベクトル



$$z = f(x, y)$$

点 $(a, b, f(a,b))$ における接平面と法線ベクトル？

点 (a, b) における Taylor 展開

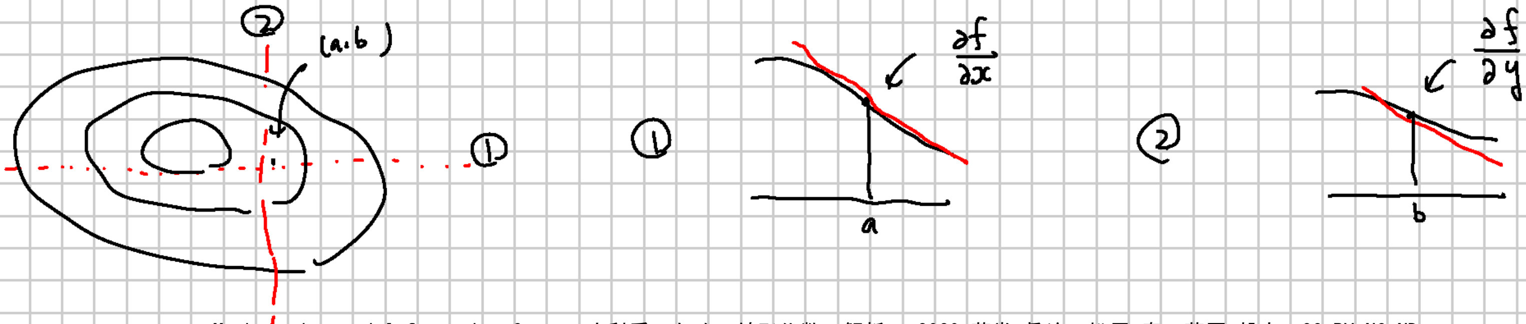
$$z = f(a,b) + f_x \cdot (x-a) + f_y \cdot (y-b) \quad + (\text{高次})$$

ムシする.

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \vec{x}_0 = \begin{pmatrix} a \\ b \\ f(a,b) \end{pmatrix} \qquad \vec{n} = \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}$$

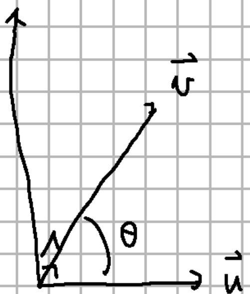
$$\boxed{\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0}$$ 接平面の方程式

$\vec{n}$: 点 $(a, b, f(a,b))$ における法線ベクトル



① $\vec{t}_1 = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix}$ ② $\vec{t}_2 = \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}$

$$\vec{n} \cdot \vec{t}_1 = \vec{n} \cdot \vec{t}_2 = 0 \quad \Leftarrow \quad \vec{n} = \vec{t}_1 \times \vec{t}_2$$



$$\vec{u} \times \vec{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \quad : \text{外積}$$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\begin{cases} (\vec{u} \times \vec{v}) \cdot \vec{u} = (\vec{u} \times \vec{v}) \cdot \vec{v} = 0 \\ |\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin\theta \end{cases}$$

▷ 陰関数 (implicit function)

$\updownarrow$

陽関数 (explicit function) $\quad y = f(x)$

$\downarrow$

$F(x, y) = 0$

例) 円を表す方程式

$$\begin{cases} x^2 + y^2 = 1 & F(x,y) = x^2 + y^2 - 1 = 0 \quad \text{陰関数} \\ y = \pm\sqrt{1 - x^2} & : \text{陽関数} \end{cases}$$

陽 ⟷ 陰

$F(x, y) = 0 \quad , \quad y = f(x)$

$$\boxed{F(x, f(x)) = 0}$$

$F=0$ は $y=f(x)$ の陰関数表示

例) $F(x,y) = x - f(y) = 0$

$y = g(x)$ 陽関数

$F(x, g(x)) = x - f(g(x)) = 0 \Rightarrow f(g(x)) = x$

$g(x)$: $f(x)$ の逆関数

▷ 陰関数の微分公式

$F(x, f(x)) = 0$

x について微分　$g_1 = x$, $g_2 = f(x)$　合成関数の微分

$$\frac{dF}{dx} = \frac{\partial F}{\partial g_1}\cdot\frac{dg_1}{dx} + \frac{\partial F}{\partial g_2}\cdot\frac{dg_2}{dx} = \underline{\frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial y}\cdot\frac{df}{dx} = 0}$$

$$\boxed{\frac{df}{dx} = -\frac{F_x}{F_y}}$$

陰関数の微分で陽関数の微分が計算できる

例)　$F(x,y) = x^2+y^2-1=0$　　　$y=\sqrt{1-x^2}=f(x)$

$F_x = 2x, \quad F_y = 2y$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}$$

$F(x,y) = x - f(y) = 0$　　　$y=g(x)$　　g は f の逆関数

$F_x = 1, \quad F_y = -\frac{df}{dy} = -f'$

$$\frac{dg}{dx} = -\frac{F_x}{F_y} = \frac{1}{f'} = \left(\frac{df}{dy}\right)^{-1}$$

逆関数の微分公式と一致.

三変数

陰関数　$F(x, y, z) = 0$

陽関数　$z = f(x, y)$

$$\boxed{F(x, y, f(x, y)) = 0}$$

例）$F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$

$$z = \sqrt{1 - x^2 - y^2} = f(x, y)$$

陰関数微分公式　$g_1 = x,\quad g_2 = y,\quad g_3 = f(x, y)$

$$\frac{\partial F(x, y, f)}{\partial x} = \frac{\partial F}{\partial g_1} \cdot \frac{\partial g_1}{\partial x} + \cdots + \frac{\partial F}{\partial g_3} \cdot \frac{\partial g_3}{\partial x}$$

$$= F_x + F_z \cdot \frac{\partial f}{\partial x} = 0 \quad \rightarrow \quad \frac{\partial f}{\partial x} = -\frac{F_x}{F_z}$$

$$\frac{\partial F(x, y, f(x, y))}{\partial y} = 0 \quad \rightarrow \quad \frac{\partial f}{\partial y} = -\frac{F_y}{F_z}$$

(＊)

陰関数定理

三変数関数 $F(x, y, z)$ が $F(a, b, c) = 0$ をみたし

$F_z(a, b, c) \neq 0$ と仮定

$\Downarrow$

(a, b) の近傍で $F(x, y, f(x, y))$ をみたす $f(x, y)$ が存在し

その偏微分が $(*)$ により与えられる